$\boxed{1}$ If $y(x)$ with $x>0$ satisfies $y' - 3\frac{y}{x} = x^2$, $y(1)=1$, then $y(e)=?$

Int. factor method: $I(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$,

so $x^{-3}y = \int x^2 \cdot x^{-3} dx = \int x^{-1} dx = \ln(x) + C$

$\rightarrow y = x^3(\ln(x) + C) = x^3 \ln(x) + Cx^3$

$y(1)=1 \rightsquigarrow 1 = 1 \cdot 0 + c \cdot 1^3 \Rightarrow c=1 \rightsquigarrow y(x) = x^3\ln(x) + x^3$

$y(e) = e^3 \cdot \underbrace{\ln(e)}_{=1} + e^3 = 2e^3$

$\boxed{2.}$ Consider diff equations (i) $\frac{dy}{dx} + 2y^4 = x^4$, with initial condition

(ii) $y^3 \frac{dy}{dx} = \sin x$, $y(0)=0$.

(iii) $\frac{dy}{dx} = \sin x$

which of these i.v.p.'s has a unique solution in some interval $a<x<b$ containing 0 ~~been~~ guaranteed by the Existence & Uniqueness theorem?

For "$\frac{dy}{dx} = f(x,y)$", we need that $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on some square including the point $(0,0)$ in the interior:

(i) $\frac{dy}{dx} = x^4 - 2y^4$ ... $f(x,y) = x^4 - 2y^4$, $\frac{\partial f}{\partial y} = -8y^3$ ... both are continuous, so OK ✓

(ii) $\frac{dy}{dx} = \frac{\sin x}{y^3}$ ... $f(x,y) = \frac{\sin x}{y^3}$, $\frac{\partial f}{\partial y} = -3\frac{\sin x}{y^4}$ ... neither is continuous when $y=0$ (~~not even defined~~ (not even defined)

$\rightarrow$ Ex & Uniq. thm does not apply

(iii) $\frac{dy}{dx} = \sin x$, ... $f(x,y) = \sin x$, $\frac{\partial f}{\partial y} = 0$ ... both are continuous, so OK ✓

$\Rightarrow$ Answer: (i) and (iii) only

3. The solution to $\frac{dy}{dx} = 4xy^2$, $y(0)=1$, is:

Separable equation $\rightarrow$ $\frac{dy}{y^2} = 4x\,dx$

$$(-1)y^{-1} = \int \frac{dy}{y^2} = \int 4x\,dx = 2x^2 + C$$

initial condition $y(0)=1$ $\rightsquigarrow$ $-1 = 0 + C$ $\rightsquigarrow$ $C = -1$

$$\rightarrow \quad -y^{-1} = 2x^2 - 1 \quad \rightsquigarrow \quad y = \frac{1}{1-2x^2} = \frac{-1}{2x^2-1}$$

4. The general solution to $\frac{dy}{dx} = (x+y)^2 - 1$ is:

RHS is of the form „$G(x+y)$" $\rightsquigarrow$ substitution $v = x+y$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dv}{dx} - 1 = v^2 - 1$$

$$\frac{dv}{dx} = v^2$$

$$\frac{dv}{v^2} = dx$$

$$-v^{-1} = \int \frac{dv}{v^2} = \int dx = x + C$$

$$-v^{-1} = x + C$$

$$y + x = v = -\frac{1}{x+C}$$

$$y = -\frac{1}{x+C} - x$$

5. The general solution to $\frac{dy}{dx} = -\frac{2xy+1}{x^2+y}$ is:

$$(x^2+y)\,dy = -(2xy+1)\,dx$$

$$\underbrace{(2xy+1)}_{M}\,dx + \underbrace{(x^2+y)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 2x \quad \Rightarrow \text{exact equation},$$

$$F(x,y) = \int (2xy+1)\,dx = x^2y + x + g(y),$$

$$x^2 + g'(y) = \frac{\partial F}{\partial y} = N(x,y) = x^2 + y$$

$$\rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} \; (+C)$$

$$\rightarrow F(x,y) = x^2y + x + \frac{y^2}{2}$$

Implicit solutions are of the form

$$\boxed{x^2y + x + \frac{y^2}{2} = C}$$

6. A ball of mass 2kg is dropped from the height = 10 m above ground in vacuum ($\Rightarrow$ no air resistance force). Assume that the only force acting on the ball is gravity and initial velocity is 0 m/sec.
The speed of the ball right before it hits the ground is:

Newton's second law $\Rightarrow$ $2 \cdot \frac{dv}{dt} = \underbrace{2 \cdot g}_{\text{gravitational force}}$

$\rightsquigarrow \frac{dv}{dt} = g$, ie $dv = g\,dt \rightsquigarrow v = \int dv = \int g\,dt = gt + C$.

initial condition $v(0)=0$: $0 = g \cdot 0 + C \Rightarrow C = 0$

$\Rightarrow v(t) = gt$, so the position function is $x(t) = \int_0^t gs\,ds = \left[\frac{gs^2}{2}\right]_0^t = \frac{gt^2}{2}$

Solve $x(t_0) = 10$ for t_0:

$$\frac{g t_0^2}{2} = 10 \rightsquigarrow t_0 = \sqrt{\frac{20}{g}}$$

Then $v(t_0)$ = velocity before impact $= g \cdot t_0 = g \cdot \sqrt{\frac{20}{g}} = \sqrt{\frac{20 g^2}{g}} = \sqrt{20 g}$

$\boxed{7}$ Let $A = \begin{bmatrix} 1 & 0 & c \\ 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & 0 \end{bmatrix}$. Which of the following is true?

(A) $A + B^T = A^T + B$: No need to compute, LHS = 2×3 matrix, RHS = 3×2 matrix, so they cannot equal

$\cancel{A+B^T = \begin{bmatrix} 1 & 0 & c \\ 1 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1+c \\ 1 & & \end{bmatrix}}$

→ not true

(B) $AB = BA$ .. similarly, AB = 2×2 matrix, BA = 3×3 matrix } cannot be equal

→ not true

(C) $(AB)^T = A^T B^T$ ~~LHS~~ $(AB)^T$ = 2×2 matrix, $A^T B^T$ = 3×3 matrix } cannot be equal

→ not true

(D) If the (1,1) entry of AB is 1, then $c = -1$;

$$AB = \begin{bmatrix} 1 & 0 & c \\ 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2+c & 0 \\ 6 & 8 \end{bmatrix}$$

$2 + c = 1$

$c = -1$

⇒ true

(E) If the (1,1)-entry of AB is 1, then $c = 1$

~~can~~ cannot be true as it contradicts the true statement (D).

→ (D) is correct.

8. If $A\vec{x} = \vec{b}$ (A matrix, $\vec{x}$, $\vec{b}$ vectors), which statement is not necessarily true?

(A) if A has 3 columns, then $\vec{x}$ has 3 entries.

correct: number of columns of A has to equal number of "rows" (ie entries) of $\vec{x}$ in order for $A\vec{x}$ to be defined.

(B) If A has 5 rows, then $\vec{b}$ has 5 entries:

correct (number of rows of A will be the number of "rows", ie entries, of the product $A\vec{x} = \vec{b}$)

(C) If $\vec{x} = \vec{0}$, then $\vec{b} = \vec{0}$ regardless of A:

correct: multiplication by zero vector always produces the zero vector.

(D) If $\vec{b} = \vec{0}$, then $\vec{x} = \vec{0}$ regardless of A:

incorrect: Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, then $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but $\vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(E) $\vec{b}$ is in the span of columns of A;

correct: if $A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & \cdots & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then

$$\vec{b} = A\vec{x} = x_1 \cdot \begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix} + \dots + x_n \begin{bmatrix} \uparrow \\ a_n \\ \downarrow \end{bmatrix} \in \mathrm{Span}\left(\begin{bmatrix} \uparrow \\ a_1 \\ \downarrow \end{bmatrix}, \begin{bmatrix} \uparrow \\ a_2 \\ \downarrow \end{bmatrix}, \dots \begin{bmatrix} \uparrow \\ a_n \\ \downarrow \end{bmatrix} \right).$$

$\rightarrow$ correct answer is D

$\boxed{9}$ Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -3 \\ 9 \\ -2 \end{bmatrix}$.

If $A\vec{x} = \vec{b}$, then $x_3 = ?$

Solve $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 2 & 3 & | & -3 \\ 1 & 5 & 3 & | & 9 \\ 0 & 1 & 2 & | & -2 \end{bmatrix} \xrightarrow{(-1)\times} \sim \begin{bmatrix} 1 & 2 & 3 & | & -3 \\ 0 & 3 & 0 & | & 12 \\ 0 & 1 & 2 & | & -2 \end{bmatrix} \xrightarrow{\text{divide by } 3} \sim \begin{bmatrix} 1 & 2 & 3 & | & -3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 1 & 2 & | & -2 \end{bmatrix} \xrightarrow{(-1)\times} \sim \begin{bmatrix} 1 & 2 & 3 & | & -3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 2 & | & -6 \end{bmatrix}$$

divide by 2

$$\sim \begin{bmatrix} 1 & 2 & 3 & | & -3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & -3 \end{bmatrix} \Rightarrow \underline{\underline{x_3 = -3}}$$

$\boxed{10}$ For which value of c does the system
$$\begin{aligned} 3x - 2y + 5z &= 1 \\ 2y + z &= 1 \\ -3x + 6y + cz &= 1 \end{aligned}$$
have infinitely many solutions?

inf. many solutions $\Leftrightarrow$ has to be consistent and have a free variable

$$\begin{bmatrix} 3 & -2 & 5 & | & 1 \\ 0 & 2 & 1 & | & 1 \\ -3 & 6 & c & | & 1 \end{bmatrix} \xrightarrow{+1\times} \sim \begin{bmatrix} 3 & -2 & 5 & | & 1 \\ 0 & 2 & 1 & | & 1 \\ 0 & 4 & c+5 & | & 2 \end{bmatrix} \xrightarrow{(-2)\times} \sim \begin{bmatrix} 3 & -2 & 5 & | & 1 \\ 0 & 2 & 1 & | & 1 \\ 0 & 0 & c+3 & | & 0 \end{bmatrix}$$

first two columns are pivot columns $\Rightarrow$ to have a free variable, third column cannot be a pivot column. That is, we need ~~c~~ $c+3=0$, so $\underline{c=-3}$ (since there is 0 on the RHS in the last row as well, we can see that the system is consistent).

11. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Which of the statements is wrong?

(A) $\vec{u} \cdot \vec{v} = -1$

$\vec{u} \cdot \vec{v} = 1 \cdot (-1) + 0 \cdot 1 = -1$ TRUE

(B) $\vec{u}^T \vec{v} = \vec{v}^T \vec{u}$:

$\vec{u}^T \vec{v} = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1 \cdot (-1) + 0 \cdot 1 = -1$ (as a 1×1 matrix if you wish),

$\vec{v}^T \vec{u} = \begin{bmatrix} -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (-1) \cdot 1 + 1 \cdot 0 = -1$ (as a 1×1 matrix if you wish)

$\Rightarrow \vec{u}^T \vec{v} = \vec{v}^T \vec{u}$ TRUE

(C) $\{\vec{u}, \vec{v}, \vec{w}\}$ are linearly independent:

$\vec{u}, \vec{v}, \vec{w}$ L.I. $\Leftrightarrow$ the vector equation $x_1 \vec{u} + x_2 \vec{v} + x_3 \vec{w} = 0$ has only the triv. solution. (i.e. only solution $x_1 = x_2 = x_3 = 0$)

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix}$ is already in row echelon form and we see that x_3 is a free variable $\Rightarrow$ non-trivial solutions $\Rightarrow$ not L.I. So FALSE

(D) $\{\vec{v}, \vec{w}\}$ is linearly dependent

TRUE: For example, a non-trivial solution to $x_2 \vec{v} + x_3 \vec{w} = 0$ is $x_2 = 2$ and $x_1 = 1$ $\left(\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \right.$ has a free variable $\left. \right)$.

(E) $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$:

TRUE: In fact, $\vec{w} = 0 \cdot \vec{u} + (-2) \cdot \vec{v}$.

$\rightsquigarrow$ Answer is C

12. $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear transformation, $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is the standard matrix for T. Let $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. ~~Consider~~ Which of the following statements are true?

(i) $T(\vec{u}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$;

$$T(\vec{u}) = A \cdot \vec{u} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ FALSE:

(ii) $T(2\vec{u} + \vec{v}) = 2T(\vec{u}) + T(\vec{v})$

TRUE (this is by definition of a linear transformation, we need to check anything)

(iii) $T(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbb{R}^2$

TRUE (this is the meaning of „A is the standard matrix for T")

$\Rightarrow$ Only the statements (ii), (iii) are true.