Quiz 1 Solution

1. Find all m such that the function $\varphi(x) = x^m$ is a solution to the equation

$$5x^2 \frac{\mathrm{d}^2 y}{(\mathrm{d}x)^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{5}y = 0 \ .$$

Solution: For $y = x^m$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1}, \quad \frac{\mathrm{d}^2 y}{(\mathrm{d}x)^2} = m(m-1)x^{m-2},$$

so plugging into the diff. equation yields

$$\begin{aligned} 5x^2 \cdot m(m-1)x^{m-2} + x \cdot mx^{m-1} + \frac{4}{5}x^m &= 0 \\ 5m(m-1)x^m + mx^m + \frac{4}{5}x^m &= 0 \\ 5m(m-1) + m + \frac{4}{5} &= 0 \\ 5m^2 - 4m + \frac{4}{5} &= 0. \end{aligned}$$

The discriminant of the resulting quadratic equation is 0, and so the unique solution to this equation is

$$m = \frac{4 \pm \sqrt{0}}{2 \cdot 5} = \frac{2}{5}.$$

Therefore the desired exponent m is $m = 2/5$.

2. Find all solutions (in implicit or explicit form) of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3e^{-y}x^2 \ .$$

Solution: This is a separable equation, so we separate the variables:

$$\begin{aligned} e^y \mathrm{d}y &= 3x^2 \mathrm{d}x \\ \int e^y \mathrm{d}y &= \int 3x^2 \mathrm{d}x \\ e^y &= x^3 + C, \end{aligned}$$

where C is any constant. This describes all the solutions implicitly. For explicit form, we may solve for y:

$$y = \ln(e^y) = \ln(x^3 + C), \quad C \in \mathbb{R}.$$

(The maximal domain of these solutions is $(-\sqrt[3]{C}, \infty)$.)