Quiz 10 Solution

1. Find the general solution to the equation (using variation of parameters)

$$y'' - 2y' + y = \frac{e^t}{t} \quad .$$

Solution: The auxiliary equation for

$$y'' - 2y' + y = 0$$

is

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)^2 &= 0 \end{aligned}$$

and so we have one double root $\lambda_1 = \lambda_2 = 1$.

Thus, the general solution to the homogeneous equation is of the form

$$y_h = c_1 e^t + c_2 t e^t \ .$$

To obtain a particular solution, we seek a solution in the from

$$y_p = v_1 e^t + v_2 t e^t.$$

To determine v_1, v_2, we have to solve the system

$$\begin{aligned} v_1' e^t + v_2' t e^t &= 0 \\ v_1' e^t + v_2'(e^t + t e^t) &= \frac{e^t}{t}. \end{aligned}$$

Subtracting the first equation from the second yields

$$\begin{aligned} v_2' e^t &= \frac{e^t}{t} \\ v_2' &= \frac{1}{t}, \end{aligned}$$

hence

$$v_2 = \int \frac{\mathrm{d}t}{t} = \ln t,$$

and plugging v_2' back into the first equation yields

$$\begin{aligned} v_1' e^t + \frac{1}{t}(t e^t) &= 0 \\ v_1' &= -1, \end{aligned}$$

so

$$v_1 = \int (-1)\mathrm{d}t = -t.$$

Thus, the obtained particular solution is of the form

$$y_p = (-t)e^t + t\ln(t)e^t$$

and the general solution is

$$y = y_p + y_h = (-t)e^t + t\ln(t)e^t + c_1e^t + c_2te^t = c_1e^t + c_2te^t + t\ln(t)e^t$$

(where the constant c_2 changed slightly in the last equality, but remained still a general constant).