Quiz 12 Solution

1. Compute the Wronskian of the vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}.$$

Solution:

$$W(\mathbf{x}_1, \mathbf{x}_2) = \begin{vmatrix} 1 & t^2 \\ t & t^3 \end{vmatrix} = t^3 - t^3 = 0.$$

(Note: You may check that $\mathbf{x}_1, \mathbf{x}_2$ are in fact linearly independent even though the Wronskian vanishes everywhere. The point is that $\mathbf{x}_2 = t^2 \cdot \mathbf{x}_1$, so for any fixed value of t, $\mathbf{x}_2(t)$ is a scalar multiple of $\mathbf{x}_1(t)$, however, the scalar changes as t changes, so as vector functions, $\mathbf{x}_2$ is not a scalar multiple of $\mathbf{x}_1$.)

2. Determine whether the vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos(t) \\ t^3 + 1 \\ \sin(t) \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ t^2 - t \\ 0 \end{bmatrix}, \ \mathbf{x}_3(t) = \begin{bmatrix} -\sin(t) \\ e^t \\ \cos(t) \end{bmatrix}$$

are linearly independent as vector functions on $(-\infty, \infty)$.

Solution: Again, we compute the Wronskian

$$W(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{vmatrix} \cos(t) & 0 & -\sin(t) \\ t^3 + 1 & t^2 - t & e^t \\ \sin(t) & 0 & \cos(t) \end{vmatrix} = (t^2 - t) \begin{vmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{vmatrix} = (t^2 - t)(\cos^2(t) + \sin^2(t)) = t^2 - t\ .$$

For $t \neq 0, 1$, the Wronskian is nonzero, so it certainly attains a nonzero value at least at some point. Thus, the vector functions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent.