Quiz 3 Solution

1. An object of mass 1 kg is dropped into the air. The air resistance force (in Newtons) is $-5v$ where v is the velocity of the object, the acceleration due to gravity is assumed to be $10\,\mathrm{m/sec}^2$. Determine the equation of the motion of the object (i.e. the equation describing the position $x(t)$).

Solution: The equation from Newton's law of motion has in this case the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - 5v$$

(where mg is the fore due to gravity and $-5v$ is the force of air resistance given in the problem statement). Plugging in $m = 1\,\mathrm{kg}$ and $g = 10\,\mathrm{m/sec}^2$ yields the linear differential equation

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} =& 10 - 5v \ , \\ \frac{\mathrm{d}v}{\mathrm{d}t} + 5v =& 10 \ .\end{aligned}$$

We can solve this either by separation of variables, or by integrating factor method. Let us do it by integrating factor. The factor is

$$\mu(t) = e^{\int 5\mathrm{d}t} = e^{5t},$$

so the solution v of the equation is

$$v(t) = \frac{\int 10e^{5t}\mathrm{d}t}{e^{5t}} = \left(2e^{5t} + C\right)e^{-5t} = 2 + Ce^{-5t} \ .$$

Plugging in the initial condition $v(0) = 0$ (meaning that at the beginning of the fall, the velocity of the dropped object is 0) yields $C = -2$. Thus,

$$v(t) = 2 - 2e^{-5t}.$$

To get the equation of the motion, i.e. to determine te position function $x(t)$, we integrate the velocity with respect to time, i.e.

$$x(t) = \int_0^t v(s)\mathrm{d}s = \int_0^t (2 - 2e^{-5s})\mathrm{d}s = \left[2s + \frac{2}{5}e^{-5s}\right]_0^t = 2t + \frac{2}{5}e^{-5t} - \frac{2}{5} \ .$$

(*Remark:* The usage of definite integration in the form as above corresponds to the choice of initial position to be $x(0) = 0$.)

2. Find a solution to the system of linear equations

$$\begin{aligned}4x_1 + 16x_2 =& 4 \\ 3x_1 + 13x_2 =& 4 \ .\end{aligned}$$

Solution: Subtracting the second equation from the first yields the system

$$\begin{aligned}x_1 + 3x_2 =& 0 \\ 3x_1 + 13x_2 =& 4 \ ,\end{aligned}$$

then subtracting the first equation from the second three times yields

$$\begin{aligned} x_1 + 3x_2 =& 0 \\ 4x_2 =& 4 \ . \end{aligned}$$

Thus, $x_2 = 1$, and plugging back into the first equation, we obtain $x_1 + 3 = 0$, that is, $x_1 = -3$.