Quiz 4 Solution

1.Find a solution to the vector equation

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} .$$

Solution: The augmented matrix of the corresponding system of linear equations is

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} .$$

Applying row operations, we may reduce the matrix as follows:

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} \sim_1 \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \sim_2 \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \end{bmatrix} \sim_3 \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix} \sim_4 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

(where
$\sim_1$=swapping the rows,
$\sim_2$=subtracting the first row from the second 2 times,
$\sim_3$=dividing the second row by -2,
$\sim_4$=subtracting the second row from the first 3 times). From the final form of the equation, we see that the (unique) solution is

$$x_1 = -1, \;\; x_2 = 2.$$

2. Determine whether the vectors

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 11 \end{bmatrix}$$

are linearly independent or not, and justify.

Solution: To decide whether the three vectors above are linearly independent is the same as to decide whether the vector equation

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 11 \end{bmatrix} = 0 \tag{1}$$

has a non-trivial solution, i.e. a solution (x_1, x_2, x_3) different from $(0, 0, 0)$. So the task really is to describe the solutions of the system of homogeneous linear equations with matrix

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 6 \\ 3 & -1 & 11 \end{bmatrix} .$$

By row operations, we may reduce the matrix as follows:

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 6 \\ 3 & -1 & 11 \end{bmatrix} \sim_0 \begin{bmatrix} 1 & -2 & -3 \\ 2 & -1 & 6 \\ 3 & -1 & 11 \end{bmatrix} \sim_1 \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 12 \\ 3 & -1 & 11 \end{bmatrix} \sim_2 \begin{bmatrix} 1 & -2 & -3 \\ 0 & 3 & 12 \\ 0 & 5 & 20 \end{bmatrix} \sim_3$$

$$\sim_3 \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 4 \\ 0 & 5 & 20 \end{bmatrix} \sim_4 \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim_5 \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

(where
$\sim_0$=multiplying the first row by (-1),
$\sim_1$=subtracting the first row from the second 2 times,
$\sim_2$=subtracting the first row from the third 3 times,
$\sim_3$=dividing the second row by 3,
$\sim_4$=subtracting the second row from the third 5 times,
$\sim_5$=adding the second row to the first 2 times).

The pivot columns are only columns 1 and 2, which means that x_3 is a free variable. That is, the original vector equation has non-trivial solutions, hence the given set of vectors is not linearly independent.

Remark for completeness: the solutions (x_1, x_2, x_3) of the equation (1) are all triples of the form

$$(-5x_3, -4x_3, x_3) \ .$$