Example

(In-formally étale $\not\Rightarrow$ weakly étale)

<u>Def</u>

- An ideal $I \subseteq A$ is <u>locally nilpotent</u> if $\forall x \in I\ \exists n : x^n = 0$
- A ring map $R \longrightarrow S$ is <u>ln-formally</u> <u>étale</u> if it has the unique lifting property w.r.t. all maps $A \xrightarrow{\text{mod } I} A/I$, I locally nilpotent

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & A \arrow[d, "\text{mod } I"] \\ S \arrow[r, "g"'] \arrow[ur, dotted, "\exists !"] & A/I \end{tikzcd}$$

Def

- $R \longrightarrow S$ is **weakly étale** if both $R \longrightarrow S$ and $S \otimes_R S \xrightarrow{\mu} S$ are flat.

Fact:

weakly étale $\Rightarrow$ ln-formally étale

Q: Is the converse true?

Ans: No.

Example

k a field, consider the map

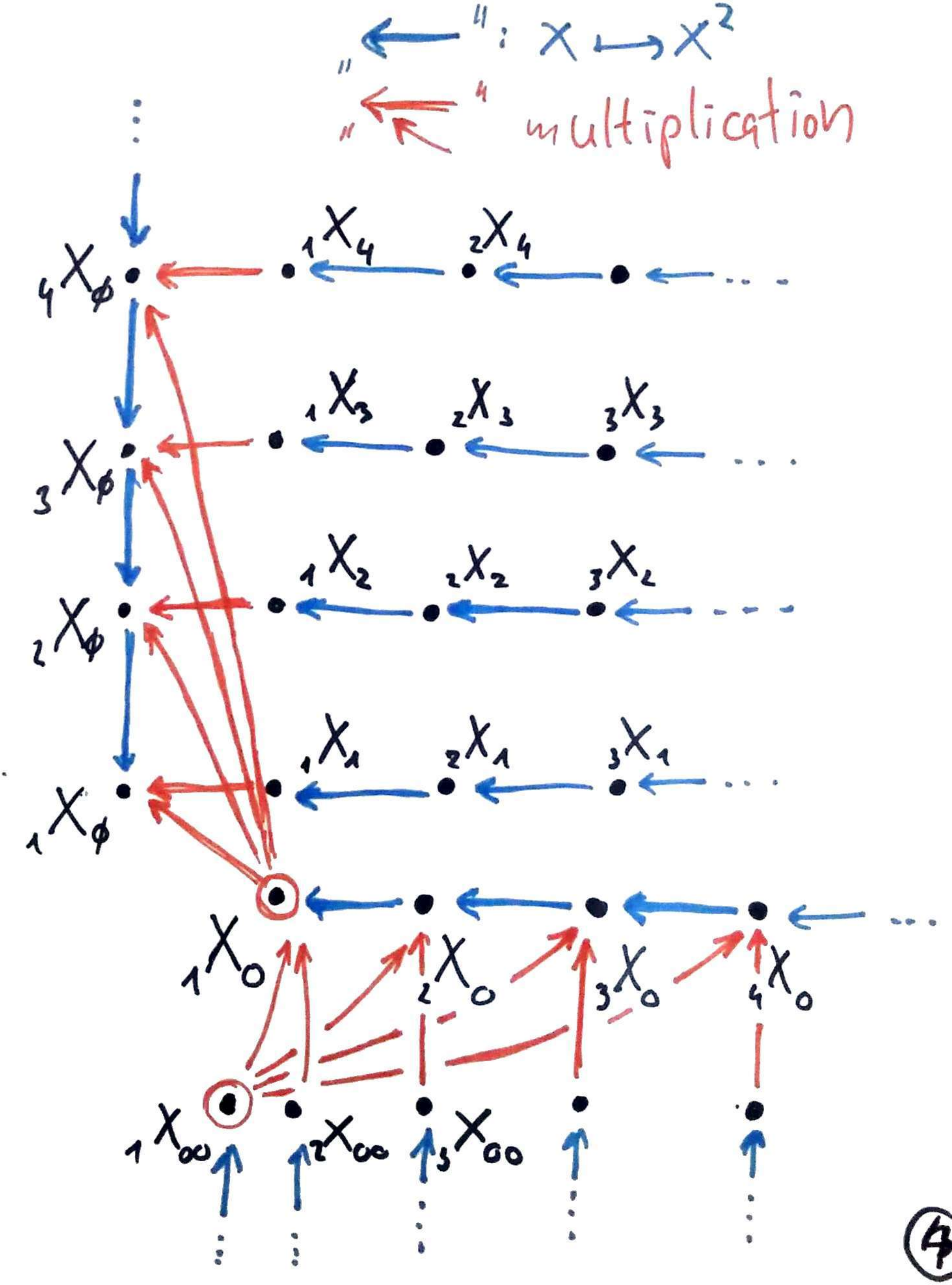
$R \longrightarrow R/J \simeq k$, where

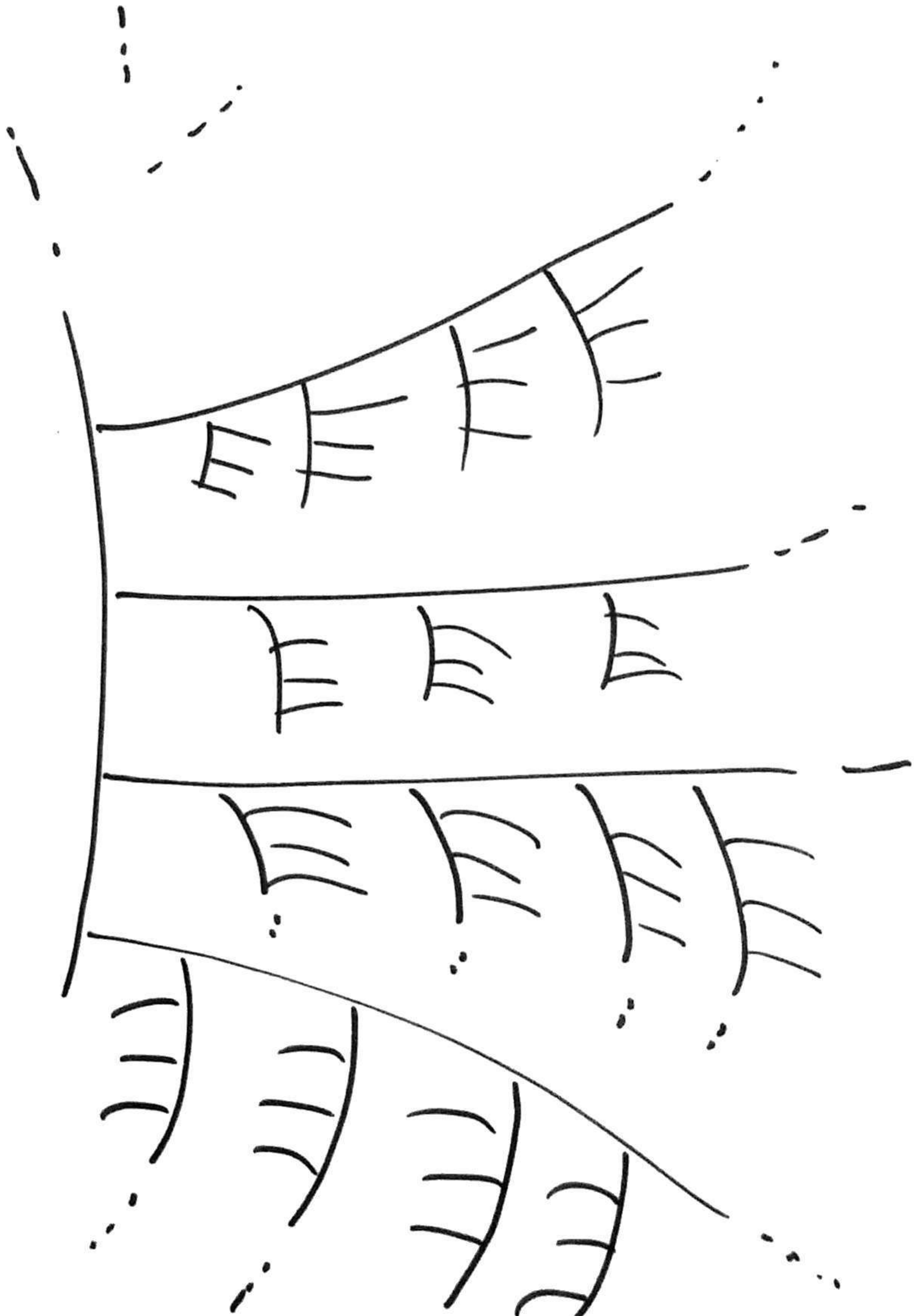
$$R = k\left[{}_aX_w \;\middle|\; \begin{matrix} a = 1,2,3,\dots \\ w = \text{word in letters } 0,1,2,\dots \end{matrix}\right] \Big/ Rel,$$

$$Rel = \left(({}_{a+1}X_w)^2 - {}_aX_w,\ ({}_1X_{w0})\cdot({}_1X_{wa}) - {}_aX_w \;\middle|\; a,w \right)$$

J = ideal of all monomials

[b. on A. Geraschenko's „form. smooth $\not\Rightarrow$ flat", MO]


"$\leftarrow$": $x \longmapsto x^2$
"$\leftarrow$" multiplication
${}_4X_\phi$ ${}_1X_4$ ${}_2X_4$
${}_3X_\phi$ ${}_1X_3$ ${}_2X_3$ ${}_3X_3$
${}_2X_\phi$ ${}_1X_2$ ${}_2X_2$ ${}_3X_2$
${}_1X_\phi$ ${}_1X_1$ ${}_2X_1$ ${}_3X_1$
${}_1X_0$ ${}_2X_0$ ${}_3X_0$ ${}_4X_0$
${}_1X_{00}$ ${}_2X_{00}$ ${}_3X_{00}$



Claim 1

$R \longrightarrow k$ is In-formally étale.

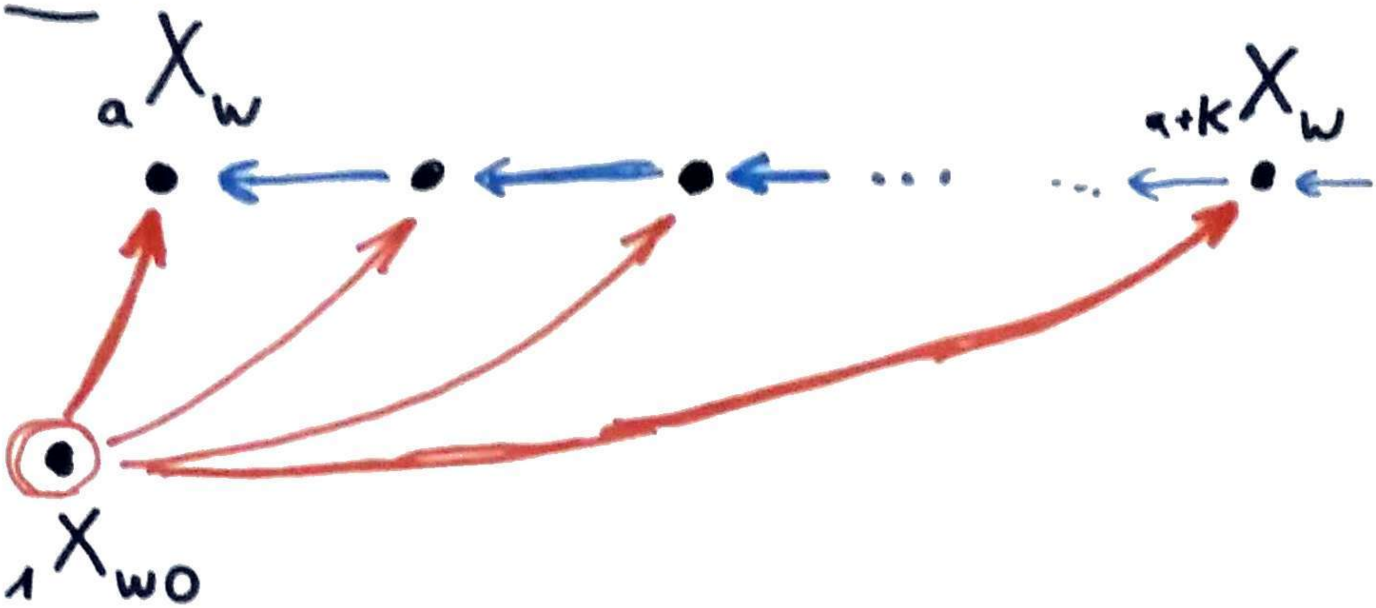
Proof:

Consider
$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ k \arrow[r] & A/I \end{tikzcd}$$

Commutativity $\Rightarrow f(J) \leq I$

Unique lifting property
$\iff f(J) = 0$

Proof:



$$f({}_1X_{w0}) \in I \Rightarrow f({}_1X_{w0})^n = 0$$

Then

$$\Rightarrow f({}_{a+k}X_w)^n = 0 \text{ for all } k \geq 0.$$

$$f({}_aX_w) = f(({}_{a+k}X_w)^{2^k}) = f({}_{a+k}X_w)^{n+(2^k-n)} =$$

$$= 0$$

$$\Rightarrow f({}_aX_w) = 0 \Rightarrow \boxed{f(J) = 0}$$

$\square$

Claim 2:

$R \longrightarrow k$ is not flat, hence not weakly étale.

Proof:

Test flatness on

$$0 \to ({}_1X_\emptyset) \to R \longrightarrow R/({}_1X_\emptyset) \to 0$$

$$\Big\downarrow \; (- \otimes_R k) = (- \otimes_R R/J):$$

$$0 \to ({}_1X_\emptyset)/J({}_1X_\emptyset) \longrightarrow k = k \to 0$$

$\Rightarrow$ enough to show: $J({}_1X_\emptyset) \subsetneq ({}_1X_\emptyset)$.

Suppose $J({}_1X_\emptyset) = ({}_1X_\emptyset)$

$\leadsto$ in $k[{}_aX_w \mid a, w]$,

we have

$$(*) \quad {}_1X_\emptyset = \underbrace{f(\underline{X})}_{\in J} \cdot {}_1X_\emptyset + \underbrace{\text{relations}(\underline{X})}_{\in \mathcal{R}el}$$

Uses fin. many variables

$\Rightarrow$ „truncate" to fin. many variables:

$$R_0 = k\left[{}_aX_w \;\middle|\; \begin{array}{l} a = 1 \dots N \\ w = \text{word on } 1 \dots N \text{ of} \\ \text{length} \le N \end{array}\right] \Big/ \mathcal{R}el_0$$

$(*) \Rightarrow J_0({}_1X_\emptyset) = ({}_1X_\emptyset)$ in R_0

Now consider

$$\varphi : R_0 \twoheadrightarrow k[X^{2^{-M}}]$$

$$_1X_\phi \longmapsto X$$

other variables $\longmapsto$ "$X^{\frac{b}{2^k}}$", determined by recursion

Then

$$\varphi(J_0) \subseteq (X^{>0}), \; \varphi(_1X_\phi) = X$$

$\Rightarrow$ $J_0(_1X_\phi) \neq (_1X_\phi)$, since

$$\varphi(J_0(_1X_\phi)) \subseteq (X^{>1}) \subsetneq (X) = \varphi((_1X_\phi))$$

$\lightning$

$\square$

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