MA 16010 Lesson 15: Related rates I

Before last time: Functions in explicit form $y = f(x)$, such as

$$y = x^2 .$$

Last time: Functions in implicit form. For example,

$$y^2 + 3xy = x^2 + y,$$

but $y = y(x)$ is "secretly" a function of x.

Today: We consider again general equations such as

$$y^2 + 3xy = x^2 + y,$$

but this time *both* $x = x(t)$ *and* $y = y(t)$ are "secretly" a function of a third variable t.

Similarly as with implicit derivatives, we can then relate the derivatives $x' = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$. (Their *rates of change* will be *related*.)

Example: A particle is moving on a circle of radius 5 centered at the origin. Its position (x, y) in the xy–plane therefore always satisfies the equation

When it passes through the point $(2, 3)$ x–coordinate changes at the rate $\frac{\mathrm{d}x}{\mathrm{d}t} = 3$ (units/second). What is the rate of change of the y–coordinate, $\frac{\mathrm{d}y}{\mathrm{d}t}$?

Summary (finding related rates): Take the derivative on both sides of the equation with respect to t. This time, use the chain rule/implicit differentiation for both $x = x(t)$ and $y = y(t)$. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[g(x)\Big] = \qquad\qquad\qquad\qquad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big[h(y)\Big] = \qquad\qquad\qquad\qquad .$$

In the end, one gets an equation involving $x, y, x' = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$. Plug in the ones that you know, and solve for the one you are trying to find.

Exercise: A radius of a circle is growing at the rate of 3 meters per second at the time when its radius is $r = 5$ m. What this the rate of change of the area of the circle at that moment?

Exercise: A water tank has a shape of a cylinder, with radius of the base 50 cm. Water escapes through a hole at the bottom of the tank at the rate 19 cm^3/s. At what rate is the water level decreasing?

Exercise: A water tank has a shape of a cone (pointing down), and the diameter fo the tank is equal to its altitude. Water escapes through a hole at the bottom of the tank at the rate 25 cm^3/s. When the water level is 125 cm, at what rate is the water level decreasing?