MA 16020 Lesson 23: Extrema of functions of two variables I

Recall (derivative tests for local max, min of a function):
Let $y = f(x)$ be a function of one variable.

1. Its local maxima and minima are among the points x_0 which satisfy

   ____ *(first derivative test).*

2. Given such a point x_0, to determine whether x_0 is a point of local maximum or minimum, we look at __________________ :

   If _________________________, then x_0 is a local minimum of f.

   If _________________________, then x_0 is a local maximum of f.

   If _________________________, then the test is inconclusive for this x_0.

   (second derivative test).

local min. local max. neither

Maxima and minima in two variables:

If $z = f(x, y)$ is a function of two variables, the folowing extrema may occur:

local min. local max. neither - saddle pt. neither - "other"

We see that in all the cases of extrema, the tangent plane to the graph is ______________________________, which can be described in terms of first partial derivatives as:

(first derivative test)

To determine what type of extreme (if any) is taking place, we use an analogous second derivative test. To perform it, we compute the *discriminant* at the given critical point:

$$D = D(x_0, y_0) =$$

If ______________________, then (x_0, y_0) is a local minimum of f.
If ______________________, then (x_0, y_0) is a local maximum of f.
If ______________________, then (x_0, y_0) is a saddle point of f.
If ___________________, then the test is inconclusive for this (x_0, y_0).

Summary (Finding extrema of functions of two variables).

1. Find all the *critical points:* Points (x, y) satisfying:

2. Compute all the second-order partial derivatives of f and $D =$

3. For a given critical point (x_0, y_0), evaluate D and f_{xx} at (x_0, y_0).

   If ______________________, then (x_0, y_0) is a local minimum of f.

   If ______________________, then (x_0, y_0) is a local maximum of f.

   If ______________________, then (x_0, y_0) is a saddle point of f.

   If ___________________, then the test is inconclusive for this (x_0, y_0).

Exercise 1. Find all the local maxima, minima and saddle points of the function

$$f(x, y) = x^3 - \frac{2}{3}y^3 - 2y^2 - 36x + 6y.$$

Exercise 2. Find all the local maxima, minima and saddle points of the function

$$f(x, y) = \frac{2}{3}y^3 + x^2 - 4yx - 10y + 6.$$

Exercise 3. Find all the local maxima, minima and saddle points of the function

$$f(x, y) = \frac{3}{2}x^4 - yx^2 + 20x^2 + \frac{1}{2}y^2 - 3.$$