Ramification bounds for mod p étale cohomology via prismatic cohomology

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Motivation & Background

Theorem (Fontaine '85)

There is no non–trivial abelian scheme over $\mathbb{Z}$. Equivalently, there is no non–zero abelian variety over $\mathbb{Q}$ with good reduction everywhere.

To prove this: let

- $K/\mathbb{Q}_p$ be a finite extension,
- $e = e(K/\mathbb{Q}_p)$ be the abs. ramification index,
- $G_K^{\mu}, \quad \mu \geq -1$ be the upper–index higher ramification subgroups of $G_K = \mathrm{Gal}(\overline{K}/K)$.

Theorem (Fontaine '85)

Let $K/\mathbb{Q}_p$ be a finite extension. Let Γ be a finite flat commutative group scheme over $\mathcal{O}_K$ that is annihilated by p^n. Then G_K^{μ} acts trivially on $\Gamma(\overline{K})$ when

$$\mu > e\left(n + \frac{1}{p-1}\right) - 1.$$

Motivation & Background

Conjecture (Fontaine '85)

Given a smooth proper $\mathcal{O}_K$–scheme X and $T = H^i_{ét}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$, G_K^{μ} acts trivially on T when

$$\mu > e\left(n + \frac{i}{p-1}\right) - 1.$$

- Special cases proved: – by Fontaine ('93) when $n = e = 1, \;\; i < p - 1,$
  – by Abrashkin ('90) when $e = 1, \;\; i < p - 1$
- Similar bounds by Hattori ('09), Caruso–Liu ('11):
  – p^n–torsion quotients of lattices in semistable $\mathbb{Q}_p$–representations
  – applies to $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p\mathbb{Z})$ when X has semistable reduction and $ie < p - 1$

Main result

Let

- $\mathcal{X}$ be a smooth and proper formal $\mathcal{O}_K$–scheme,
- $\mathbb{C}_K = \widehat{\overline{K}}$,
- $T = H^i_{\text{ét}}(\mathcal{X}_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z})$, viewed as a representation of G_K.

Theorem (Č.)

Set

$$\alpha = \left\lfloor \log_p \left(\max \left\{ \frac{ip}{p-1}, \frac{(i-1)e}{p-1} \right\} \right) \right\rfloor + 1 \quad \textit{and} \quad \beta = \frac{1}{p^\alpha} \left(\frac{iep}{p-1} - 1 \right).$$

Then for every $\mu > e\alpha + \max\{\beta, e/(p-1)\}$, G_K^μ *acts trivially on* T.

Main result

Corollary (non–optimal, but more tractable)

G_K^{μ} acts trivially on $T = H^i_{\acute{e}t}(\mathcal{X}_{\mathbb{C}_K}, \mathbb{Z}/p\mathbb{Z})$ when

- $e \leq p$ *and* $\mu > e\left(\left\lfloor \log_p\left(\frac{ip}{p-1}\right)\right\rfloor + 1\right) + e,$
- $e > p$ *and* $\mu > e\left(\left\lfloor \log_p\left(\frac{ie}{p-1}\right)\right\rfloor + 1\right) + p,$
- $i = 1$ *(e, p arbitrary) and* $\mu > e\left(1 + \frac{1}{p-1}\right).$

Some comparisons:

- $ie < p - 1$ or $i = 1$: agrees with Hattori, Caruso–Liu
- $e = 1$ and $i < p - 1$: Fontaine, Abrashkin are slightly stronger
- Bound of Caruso ('13) a posteriori applies; which bound is stronger depends on K

Fontaine's condition (P_m)

- Fix L/K finite Galois extension.

$$(P_m^{L/K}) \qquad \begin{array}{c} \text{For every alg. extension } E/K : \\ \exists\ \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}_E^{>m} \ \text{ over } \mathcal{O}_K \ \Rightarrow\ \exists\ L \hookrightarrow E \ \text{ over } K \end{array}$$

- here $\mathfrak{a}_E^{>m} = \{x \in \mathcal{O}_E \mid v_K(x) > m\}$.

Proposition (Fontaine, Yoshida)

$$\inf\{\mu \mid \mathrm{Gal}(L/K)^{\mu-1} = 1\} = \inf\{m \mid (P_m^{L/K}) \text{ } \textit{holds}\}.$$

"Meta–strategy":

- To the rep. T, attach associated modules (+ extra structure).
- Encode and prove (P_m) in terms of the modules. (L = splitting field of T or variant)

Breuil–Kisin(–Fargues) modules

- $\mathcal{O}_K \hookrightarrow \mathcal{O}_{\mathbb{C}_K}$ + choice of $\pi \in K$ uniformizer, $\pi_s = \pi^{1/p^s}$, $s \geq 0$ determine

$$\mathfrak{S} = W(k)[[u]] \longrightarrow A_{\mathrm{inf}} = W(\mathcal{O}^\flat_{\mathbb{C}_K})$$
$$u \longmapsto [\underline{\pi}], \quad \underline{\pi} = (\pi_s)_s \in \mathcal{O}^\flat_{\mathbb{C}_K}$$

- A **Breuil–Kisin (BK) module** is a fin.-gen. $\mathfrak{S}$–module M_{BK} + a semilinear operator $\varphi : M_{\mathrm{BK}} \to M_{\mathrm{BK}}$ satisfying certain invertibility condition
- When M_{BK} is a BK module, the module $M_{\mathrm{inf}} = M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ is called a **Breuil–Kisin–Fargues (BKF) module**. If M_{inf} has G_K–action, it is a **BKF G_K–module**.

New input: Prismatic cohomology

- Bhatt–Morrow–Scholze ('16, '18), Bhatt–Scholze ('19)
- $\mathfrak{X}$ a smooth formal $\mathcal{O}_K$–scheme $\rightsquigarrow$ **Breuil–Kisin (prismatic) cohomology** $R\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S})$
  - $H^i_{\Delta}(\mathfrak{X}, \mathfrak{S})$ is a BK module when $\mathfrak{X}$ is proper
- $\mathfrak{Y}$ a smooth formal $\mathcal{O}_{\mathbb{C}_K}$–scheme $\rightsquigarrow$ **A_{inf}–(prismatic) cohomology** $R\Gamma_{\Delta}(\mathfrak{Y}/A_{\mathrm{inf}})$
  - $H^i_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}, A_{\mathrm{inf}})$ is a BKF G_K–module when $\mathfrak{X}$ is proper
  - $H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_K}, \mathbb{Z}_p) = (H^i_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}, A_{\mathrm{inf}}) \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_K^\flat))^{\varphi=1}$ as G_K–modules
- p^n–torsion version (Li–Liu '20): $R\Gamma_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) = R\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \otimes^{\mathrm{L}} \mathbb{Z}/p^n\mathbb{Z}$, same for A_{inf}
  - $H^i_{\Delta,n}(\mathfrak{X}, \mathfrak{S})$ are BK modules, $H^i_{\Delta,n}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}, A_{\mathrm{inf}})$ are BKF G_K–modules when $\mathfrak{X}$ is proper
  - $H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_K}, \mathbb{Z}/p^n\mathbb{Z}) = (H^i_{\Delta,n}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}, A_{\mathrm{inf}}) \otimes_{A_{\mathrm{inf}}} W_n(\mathbb{C}_K^\flat))^{\varphi=1}$ as G_K–modules

The conditions (Cr_s)

- Set $G_s = \mathrm{Gal}(\overline{K}/K(\pi_s))$, $G_\infty = \bigcap_s G_s$
- Fix $M_{\mathrm{BK}} = H^i_{\Delta,1}(\mathfrak{X}/\mathfrak{S})$, $M_{\mathrm{inf}} = H^i_{\Delta,1}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$, and set

$$T^{\mathrm{BK}} = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, \mathcal{O}^\flat_{\mathbb{C}_K}) \simeq T^\vee|_{G_\infty}, \quad T^{\mathrm{inf}} = \mathrm{Hom}_{A_{\mathrm{inf}},\varphi}(M_{\mathrm{inf}}, \mathcal{O}^\flat_{\mathbb{C}_K}) \simeq T^\vee.$$

- Idea: Mutate these enough to verify Fontaine's property (P_m)
- Need to relate the G_s–actions on T^{BK} and T^{inf} by a statement of the type

$$\forall g \in G_s : (g-1)M_{\mathrm{BK}} \subseteq \mathfrak{a}^{>c} M_{\mathrm{inf}}$$

- Reminiscent of a crystallinity criterion of Gee–Liu ('19):

(Cr_0) $$\forall g \in G_K : \quad (g-1)M_{\mathrm{BK}} \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]M_{\mathrm{inf}}$$

The conditions (Cr_s)

$\check{C}^\bullet_{\mathrm{BK}}, \check{C}^\bullet_{\mathrm{inf}}$ = Čech–Alexander complexes modelling $\mathrm{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}), \mathrm{R}\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}})$

There is a decreasing sequence of ideals $I_s \subseteq A_{\mathrm{inf}},\ s \geq 0$ such that:

Theorem (Č.)

For every i and every $s \geq 0$, one has $\check{C}^i_{\mathrm{BK}} \widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}} = \check{C}^i_{\mathrm{inf}}$ and

$$(\mathrm{Cr}_s) \qquad \forall g \in G_s : \quad (g-1)\check{C}^i_{\mathrm{BK}} \subseteq I_s \check{C}^i_{\mathrm{inf}}.$$

Consequently,

(1) *For every i, the cohomology groups satisfy*

$$(\mathrm{Cr}_0) \qquad \forall g \in G_K : \quad (g-1)H^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]H^i_{\Delta}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}).$$

(2) *For every i and every pair of integers s, n with $s+1 \geq n \geq 1$, one has*

$$\forall g \in G_s : \quad (g-1)H^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) \subseteq ([\underline{\varepsilon}^{1/p}]-1)[\underline{\pi}]^{p^{s-n+1}}H^i_{\Delta,n}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}/A_{\mathrm{inf}}).$$

Thank you!