MA 16020 Quiz 5 (Lessons 9-10)

1. Find the general solution to the differential equation

$$x^3y' + x^2y + x = 0\ , \quad x > 0.$$

2. A tank initially contains 100 gallons of pure water. A brine containing 2 pounds of salt per gallon flows into the tank at the rate 2 gallons per minute, and the well-stirred mixture flows out of the tank at the rate 1 gallon per minute. How much salt is in the tank after 10 minutes? Round the answer (in pounds) to two decimal places.

Solution:

1.

$$x^3y' + x^2y + x = 0 \qquad \Rightarrow \qquad y' + \frac{1}{x}y = -\frac{1}{x^2}; \quad u(x) = e^{\int \frac{1}{x}\mathrm{d}x} = e^{\ln(x)} = x$$

$$y = \frac{1}{x} \cdot \int x \cdot \left(-\frac{1}{x^2}\right) \mathrm{d}x = \frac{1}{x} \cdot \int \left(-\frac{1}{x}\right) \mathrm{d}x = \frac{1}{x} \cdot (-\ln(x) + C) = \frac{C}{x} - \frac{\ln(x)}{x}$$

2. $A(t)$ = amount of salt after t minutes; $A(0) = 0$
Volume of mixture in the tank: $V(t) = 100 + t$

Rate of salt going in: (2 gal/min) $\cdot$ (2 lb/gal) = 4 lb/min

Rate of salt going out: (1 gal/min) $\cdot$ ($\frac{A(t)}{100+t}$ lb/gal) = $\frac{A}{100+t}$ lb/min

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 4 - \frac{A}{100+t},$$

First we solve the equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{A}{100+t} = 4 \qquad \rightsquigarrow \qquad u(t) = e^{\int \frac{\mathrm{d}t}{100+t}} = e^{\ln(100+t)} = 100 + t$$

$$A(t) = \frac{1}{100+t} \cdot \int 4(100+t)\mathrm{d}t = \frac{1}{100+t} \cdot \left(400t + 2t^2 + C\right)$$

To determine C, we use the initial condition $A(0) = 0$:

$$0 = \frac{1}{100+0} \cdot \left(400 \cdot 0 + 2 \cdot 0^2 + C\right) \qquad \Rightarrow \qquad C = 0$$

So we have

$$A(t) = \frac{1}{100+t} \cdot \left(400t + 2t^2\right),$$

hence

$$A(10) = \frac{1}{100+10} \cdot \left(400 \cdot 10 + 2 \cdot 10^2\right) = \frac{4200}{110} \approx 38.18 \text{ lb}$$

Thus, after 10 minutes there will be 38.18 pounds of salt in the tank.