The purpose of this note is to supply detailed proofs of existence and basic properties of "π-typical Witt vectors", that is, an analogue of p-typical Witt vectors when the prime \( p \) is replaced by a uniformizer \( \pi \) of a local number field, as introduced in [1], and further described in [2].

The basic setup is as follows. Let \( E/\mathbb{Q}_p \) be a finite extension, \( [E : \mathbb{Q}_p] = ef \) the ramification and inertia indices, let \( \mathcal{O}_E \) be the ring of integers, \( \pi \in \mathcal{O}_E \) a uniformizer, and \( \kappa_E \simeq \mathbb{F}_q \) the residue field where \( q = p^f \).

Denote by \( U : \mathcal{O}_E-\text{Alg} \to \text{Set} \) the forgetful functor, \( U^\omega : \mathcal{O}_E-\text{Alg} \to \text{Set} \) the functor \( A \mapsto A^\omega \), \( (w_{\pi,n} =) w_n : U^\omega \implies U \) the natural transformation given by

\[
(w_{n,A}) : A^\omega \to A, \quad (a_k)_{k\geq 0} \mapsto \sum_{k=0}^{n} \pi^k a_k^{q^{n-k}}
\]

(and identify \( w_n \) with the polynomial \( w_n(X) = \sum_{k=0}^{n} \pi^k X_k^{q^{n-k}} \)). The collection of all \( w_n^\prime s \) assemble to a natural transformation \( (w_{\pi} =) w : U^\omega \implies U^\omega \). Denote by \( \text{Id}^\omega : \mathcal{O}_E-\text{Alg} \to \mathcal{O}_E-\text{Alg} \) the functor \( A \to A^\omega \) (with the product ring structure on \( A^\omega \)).

**Proposition 1.** There exists a unique functor \( W_{E,\pi} = W_E : \mathcal{O}_E-\text{Alg} \to \mathcal{O}_E-\text{Alg} \) fitting into the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_E-\text{Alg} & \xrightarrow{U^\omega} & \text{Set} \\
\downarrow W_E & & \downarrow U \\
\mathcal{O}_E-\text{Alg} & \xrightarrow{U} & \text{Set}
\end{array}
\]

such that \( w : U^\omega \implies U^\omega \) is a natural transformation \( w : W_E \implies \text{Id}^\omega \).

**Remark 2.** When one takes \( E = \mathbb{Q}_p \) and \( \pi = p \), \( W_E \) recovers the standard \( p \)-typical vectors.

**Lemma 3.** Given a polynomial \( \Phi(X,Y) \in \mathcal{O}_E[X,Y] \), there exist polynomials \( \Phi_n(X,Y) \in \mathcal{O}_E[X_0, X_1, \ldots X_n, Y_0, Y_1, \ldots, Y_n] \) such that \( \forall n \geq 0, \Phi(w_n(X),w_n(Y)) = w_n(\Phi(X,Y)) \), i.e.

\[
\Phi \left( \sum_{k=0}^{n} \pi^k X_k^{q^{n-k}}, \sum_{k=0}^{n} \pi^k Y_k^{q^{n-k}} \right) = \sum_{k=0}^{n} \pi^k \Phi_k(X,Y)^{q^{n-k}}.
\]
Proof. The polynomials $\phi_n(X,Y) \in E[X,Y]$ do exist and are necessarily unique. Indeed, clearly $\Phi_0(X,Y) = \Phi(X,Y)$, and $\Phi_n(X,Y)$ is obtained from $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$ by

$$\Phi_n(X,Y) = \frac{1}{n!} \left( \Phi \left( \sum_{k=0}^{n} \pi^k X^{q^{-k}}, \sum_{k=0}^{n} \pi^k Y^{q^{-k}} \right) - \sum_{k=0}^{n-1} \pi^k \Phi_k(X,Y)^{q^{-k}} \right).$$

What remains is to show integrality of the coefficients of each $\Phi_n$.

We proceed by induction. Suppose that all the polynomials $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$ have integral coefficients. Let $\equiv$ denote the congruence mod $\pi^n$. Then

$$\sum_{k=0}^{n} \pi^k X^{q^{-k}} = \sum_{k=0}^{n} \pi^k X^{q^{-k}} + \pi^n X_n \equiv \sum_{k=0}^{n} \pi^k X^{q^{-k}},$$

hence

$$\Phi \left( \sum_{k=0}^{n} \pi^k X^{q^{-k}}, \sum_{k=0}^{n} \pi^k Y^{q^{-k}} \right) \equiv \Phi \left( \sum_{k=0}^{n-1} \pi^k X^{q^{-k}}, \sum_{k=0}^{n-1} \pi^k Y^{q^{-k}} \right) = \Phi \left( \sum_{k=0}^{n-1} \pi^k (X^q)^{q^{-k-1}}, \sum_{k=0}^{n-1} \pi^k (Y^q)^{q^{-k-1}} \right) \equiv \sum_{k=0}^{n-1} \pi^k \Phi_k(X^q,Y^q)^{q^{-k-1}}.$$

Since the polynomials $\Phi_k$ in the last expression are all integral by induction hypothesis, we have that $\Phi_k(X^q,Y^q)$ is congruent to $\Phi_k(X,Y)^q$ modulo $\pi$, i.e.

$$\Phi_k(X^q,Y^q) = \Phi_k(X,Y)^q + \pi A_k, \ A_k \in \mathcal{O}_E[X,Y].$$

Thus, we obtain

$$\Phi \left( \sum_{k=0}^{n} \pi^k X^{q^{-k}}, \sum_{k=0}^{n} \pi^k Y^{q^{-k}} \right) \equiv \sum_{k=0}^{n-1} \pi^k (\Phi_k(X,Y)^q + \pi A_k)^{q^{-k-1}}.$$

Now by binomial theorem, we have

$$\pi^k (\Phi_k(X,Y)^q + \pi A_k)^{q^{-k-1}} = \pi^k \Phi_k(X,Y)^{q^{-k}} + \sum_{j=1}^{q^{k-1}} \left( \begin{array}{c} q^{n-1-k} \\ j \end{array} \right) \pi^{j+k} B_k, \ B_k \in \mathcal{O}_E[X,Y],$$

and

$$\text{val}_p \left( \left( \begin{array}{c} q^{n-1-k} \\ j \end{array} \right) \right) = f(n-1-k) - \text{val}_p(j) \geq n - 1 - k - (j - 1) = n - k - j,$$

hence

$$\text{val}_e \left( \left( \begin{array}{c} q^{n-1-k} \\ j \end{array} \right) \pi^{j+k} \right) \geq e(n - k - j) + j \geq (n - k - j) + j = n.$$

Thus, we have that $(\Phi_k(X,Y)^q + \pi A_k)^{q^{-k-1}} \equiv \Phi_k(X,Y)^q$, and we conclude that

$$\Phi \left( \sum_{k=0}^{n} \pi^k X^{q^{-k}}, \sum_{k=0}^{n} \pi^k Y^{q^{-k}} \right) \equiv \sum_{k=0}^{n-1} \pi^k (\Phi_k(X,Y)^q)^{q^{-k-1}} = \sum_{k=0}^{n-1} \pi^k \Phi_k(X,Y)^{q^{-k}},$$

as desired. \hfill \Box
Lemma 4. Let \( x \in \mathcal{O}_E \). Then there are unique elements \( s_n(x) \in \mathcal{O}_E \) with the property

\[
(w_n(z(x)) = \sum_{k=0}^{n} \pi^k s_k(x)q^{n-k} = x \forall n.
\]

Moreover, if \( \Phi, \Phi_n \) are as in Lemma 3, then we have

\[
\forall x, y \in \mathcal{O}_E \forall n \geq 0 : s_n(\Phi(x, y)) = \Phi_n(z(x), z(y)).
\]

Proof. For \( x \in \mathcal{O}_E \), the elements \( s_n(x) \in \mathcal{O}_E \) necessarily have to be given recursively as follows:

\[
s_0(x) = x,
\]

\[
s_n(x) = \frac{1}{\pi^n} \left( x - \sum_{k=0}^{n-1} \pi^k s_k(x)q^{n-k} \right).
\]

The claim is that these are well-defined elements of \( \mathcal{O}_E \), which again boils down to inductively verifying the congruence

\[
x \equiv \sum_{k=0}^{n-1} \pi^k s_k(x)q^{n-k} \pmod{\pi^n}.
\]

For \( n = 1 \), this is clear since \( x \equiv x^q \pmod{\pi} \) for all \( x \in \mathcal{O}_E \). To check it for a general \( n \), note that we have

\[
(*) : = \sum_{k=0}^{n-1} \pi^k s_k(x)q^{n-k} = \sum_{k=0}^{n-1} \pi^k s_k(x)^q q^{n-1-k},
\]

where \( s_k(x)^q \equiv s_k(x) \pmod{\pi} \), i.e. \( s_k(x)^q = s_k(x) + \pi t, t \in \mathcal{O}_E \), and after binomial expansion of \( s_k(x)^q q^{n-1-k} \), the same estimates as in the proof of Lemma 3 show that

\[
\pi^k (s_k(x) + \pi t) q^{n-1-k} \equiv \pi^k (s_k(x))^q q^{n-1-k} \pmod{\pi^n}.
\]

Thus, we obtain

\[
(*) \equiv \sum_{k=0}^{n-1} \pi^k s_k(x)q^{n-1-k} = x \pmod{\pi^n}.
\]

What remains is to verify the last identity, which holds for \( n = 0 \) and follows immediately by induction from the established relations:

\[
s_n(\Phi(x, y)) = \frac{1}{\pi^n} \left( \Phi(x, y) - \sum_{k=0}^{n-1} \pi^k s_k(\Phi(x, y))q^{n-k} \right) = \frac{1}{\pi^n} \left( \Phi \left( \sum_{k=0}^{n} \pi^k s_k(x)q^{n-k}, \sum_{k=0}^{n} \pi^k s_k(y)q^{n-k} \right) - \sum_{k=0}^{n-1} \pi^k \Phi_k(z(x), z(y))q^{n-k} \right) = \Phi_n(z(x), z(y)).
\]

\( \square \)
Proof of Proposition 7. To describe $W_E$ on an object $A \in \mathcal{O}_E$-Alg is to endow $A^\omega$ with an $\mathcal{O}_E$-algebra structure that makes $w_A : A^\omega \to A^\omega$ a map of $\mathcal{O}_E$-algebras (where the ring structure on the codomain is component-wise).

To specify the ring structure, consider $\Phi_n(X,Y)$ obtained from $\Phi(X,Y) = X + Y$, and $\Psi_n(X,Y)$ obtained from $\Psi(X,Y) = XY$ by Lemma 3. Then in $W_E(A)$, set

$$
(a_n)_n + (b_n)_n := (\Phi_n(a,b))_n,
$$

$$
(a_n)_n \cdot (b_n)_n := (\Psi_n(a,b))_n.
$$

Finally, set $1 = (1,0,0,\ldots)$ and $0 = (0,0,0,\ldots)$.

Note that the above ring structure has functoriality in the sense that given an $\mathcal{O}_E$-algebra map $f : A \to B$, the induced map $W_E(f) = f^\omega : W_E(A) \to W_E(B)$ (defined by $f$ on each component) respects the above operations, so it is a morphism of algebraic structures with ring signatures, and it will be a ring homomorphism once we show that $W_E(A), W_E(B)$ are rings.

It is also clear from Lemma 3 (and the def. of $w$) that $w : W_E(A) \to A^\omega$ is a homomorphism of algebraic structures with ring signatures.

If $A$ is a free $\mathcal{O}_E$-algebra, then we have $A \subseteq B := A[1/\pi]$, and it is easy to see that $w_B : W_E(B) \to B^\omega$ is an isomorphism, with inverse given as follows:

$$
(w^{-1})_0(X) := X_0,
$$

$$
(w^{-1})_n(X) := \frac{1}{\pi^n} \left( X_n - \sum_{k=0}^{n-1} \pi^{k(w^{-1})_k(X)} q^{n-k} \right).
$$

Thus, it follows in this case that $W_E(B)$ is a commutative ring. Since $W_E(A) \subseteq W_E(B)$ is a substructure in the ring signature, $W_E(A)$ is a commutative ring as well.

Furthermore, note that checking for an $\mathcal{O}_E$-algebra $A$ whether $W_E(A)$ with the operations defined above is a commutative ring amounts to checking whether certain polynomial identities (for $\Phi_n$’s and $\Psi_n$’s) hold on $A$. But by the arguments above, the very same polynomial identities hold on all free $\mathcal{O}_E$-algebras, so on any $\mathcal{O}_E$-algebra $A$. Thus, we conclude that $W_E(A)$ is a commutative ring for all $\mathcal{O}_E$-algebras $A$.

Finally, we specify the $\mathcal{O}_E$-algebra structure on $W_E(A)$. For $A = \mathcal{O}_E$, we have a map $s : \mathcal{O}_E \to W_E(A)$, which is a ring homomorphism by Lemma 4 (and the fact that $s(1) = 1$). This gives an $\mathcal{O}_E$-algebra structure to $W_E(\mathcal{O}_E)$, and $W_E(A)$ of an arbitrary $\mathcal{O}_E$-algebra $\mathcal{O}_E \xrightarrow{s} A$ receives an $\mathcal{O}_E$-algebra structure by

$$
\mathcal{O}_E \xrightarrow{s} W_E(\mathcal{O}_E) \xrightarrow{W_E(s)} W_E(A).
$$

This turns tautologically all maps $W_E(f) : W_E(A) \to W_E(B)$ and all $w_A : W_E(A) \to A^\omega$ into $\mathcal{O}_E$-algebra homomorphisms.

Proposition 5. In the above setup,

1. The functor $W_{E,\pi}$ is unique.

2. Given a second uniformizer $\pi'$, there is a natural isomorphism $\alpha_{\pi',\pi} : W_{E,\pi} \xrightarrow{\cong} W_{E,\pi}$ that is also natural in $\pi$, i.e. there is a functor from the groupoid (setoid) of uniformizers of $E$ to $\text{Fun}(\mathcal{O}_E$-Alg, $\mathcal{O}_E$-Alg), sending $\pi$ to $W_{E,\pi}$ and $u = \pi'/\pi : \pi \to \pi'$ to $\alpha_{\pi',\pi}$.

4
Lemma 6. Let \( \pi, \pi' \) be two uniformizers of \( E \). There is a unique family of polynomials \( \alpha_n(X) = \alpha_{\pi', \pi, n}(X) \in \mathcal{O}_E[X_0, \ldots, X_n] \), \( n \geq 0 \), such that for all \( n \),

\[
\sum_{k=0}^{n} (\pi')^k \alpha_k(X) q^{n-k} = \sum_{k=0}^{n} \pi^k X_k q^{n-k}.
\]

Additionally, the polynomials \( \alpha \) satisfy

\[
\alpha_{\pi, \pi, n}(X) = X_0 \quad \forall n,
\]

\[
\alpha_{\pi', \pi', n}(\alpha_{\pi', \pi, n}(X)) = \alpha_{\pi', \pi, n}(X) \quad \forall n.
\]

Moreover, if \( \Phi_{\pi, n}(X, Y), \Phi_{\pi', n}(X, Y) \) are polynomials as in Lemma 3 and for \( x \in \mathcal{O}_E, s_{\pi, n}(x) \) and \( s_{\pi', n}(x) \) are elements as in Lemma 4, then we have

\[
\Phi_{\pi', n}(\alpha(X), \alpha(Y)) = \alpha_n(\Phi_E(X, Y))
\]

and

\[
\alpha_n(s_E(x)) = s_{\pi', n}(x).
\]

Proof. As in Lemmas 3 and 4, we necessarily have \( \alpha_0(X_0) = X_0 \) and

\[
\alpha_n(X) = \frac{1}{(\pi')^n} \left( \sum_{k=0}^{n} \pi^k X_k q^{n-k} - \sum_{k=0}^{n-1} (\pi')^k \alpha_k(X) q^{n-k} \right),
\]

so we again only need to check

\[
\sum_{k=0}^{n} \pi^k X_k q^{n-k} \equiv \sum_{k=0}^{n-1} (\pi')^k \alpha_k(X) q^{n-k} \quad (\text{mod } (\pi')^n)
\]

by induction on \( n \). We have

\[
\sum_{k=0}^{n-1} (\pi')^k \alpha_k(X) q^{n-k} = \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(X q) + \pi't_k) q^{n-k} = \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(X q) + \pi't_k) q^{n-1-k},
\]

t \in \mathcal{O}_E,

and again the binomial expansion of \((\alpha_k(X q) + \pi't_k) q^{n-1-k}\) and the usual estimates thus show that

\[
\sum_{k=0}^{n-1} (\pi')^k \alpha_k(X) q^{n-k} \equiv \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(X q) + \pi't_k) q^{n-1-k} \equiv \sum_{k=0}^{n-1} (\pi')^k (\alpha_k(X q)) q^{n-1-k} \quad (\text{mod } (\pi')^n).
\]

By the induction hypothesis, the right-hand side equals

\[
\sum_{k=0}^{n-1} \pi^k (X_k q) q^{n-1-k} = \sum_{k=0}^{n-1} \pi^k (X_k) q^{n-k} = \sum_{k=0}^{n} \pi^k (X_k) q^{n-k} \quad (\text{mod } (\pi')^n),
\]

so we are done.
The identity $\alpha_{\pi,\pi,n}(X) = X_n$ follows from the uniqueness of $\alpha_{\pi,\pi,n}(X)$ since $\alpha_n(X) = X_n$ obviously works. Similarly, the identity $\alpha_{\pi',\pi,\pi',n}(\alpha_{\pi',\pi}(X)) = \alpha_{\pi',\pi,\pi',n}(X)$ follows from uniqueness since $w_{\pi',n}(\alpha_{\pi',\pi}(X)) = w_{\pi',n}(X)$.

The additional identities follow from uniqueness of $\Phi_{\pi',n}(\alpha(X), \alpha(Y))$, $s_{\pi',n}(x)$ with respect to their defining properties together with the identities $w_{\pi',n}(\alpha(\Phi_{\pi,\pi}(X, Y))) = w_{\pi',n}(\alpha(\Phi_{\pi,\pi}(X, Y)))$

and

$x = w_{\pi,n}(s_{\pi}(x)) = w_{\pi,n}(\alpha(s_{\pi}(x)))$.

Proof of Proposition 3. Since $w_{A,\pi}$ is injective for all subalgebras of algebras of the form $\mathcal{O}_E[[X_i]]/[1/\pi]$, the algebra structure $W_E(A)$ is uniquely determined on these algebras, in particular on free $\mathcal{O}_E$-algebras. Given a general $\mathcal{O}_E$-algebra $A$, let $f : B \to A$ be a surjective $\mathcal{O}_E$-algebra homomorphism. Then the functor $U\omega$ determines that $W_{E,\pi}(f)$ is necessarily $f\omega : W_{E,\pi}(B) \to W_{E,\pi}(A)$, and this is obviously surjective. Thus, to make this a ring homomorphism there is only one choice of a ring structure on $W_{E,\pi}(A)$. This proves (1).

To prove (2), set $\alpha_{\pi',\pi} W_{E,\pi} \Rightarrow W_{E,\pi'}$ to be the map polynomially given by polynomials from Lemma 6 that is,

$\alpha_{\pi',\pi} : W_{E,\pi}(A) \longrightarrow W_{E,\pi'}(A)$

$\quad \quad \quad \quad (a_k)_{k \geq 0} \longmapsto (\alpha_k(\{a_l\}_{l \geq k}))_{k \geq 0}$.

Then the relations with $\Phi_n$'s and $s_n$'s from Lemma 6 immediately translate to the fact that each $\alpha_{\pi',\pi,A}$ is an $\mathcal{O}_E$-algebra homomorphism. Furthermore, Lemma 6 implies that $\alpha_{\pi',\pi} \circ \alpha_{\pi',\pi} = \alpha_{\pi',\pi}$, proving the claimed functoriality as well as the fact that $\alpha_{\pi',\pi}$ is invertible with the inverse given by $\alpha_{\pi,\pi'}$. 

References
