1. Solve the initial value problem

$$\frac{dy}{dx} = \frac{2x(y-1)}{x^2+3}, \quad y(1)=9 \; ?$$

Separation of variables: $\frac{dy}{y-1} = \frac{2x\,dx}{x^2+3}$

$$\ln(y-1) = \int \frac{2x\,dx}{x^2+3} = \left| \begin{array}{l} t = x^2+3 \\ dt = 2x\,dx \end{array} \right| = \int \frac{dt}{t} = \ln t + C$$

$$= \ln(x^2+3) + C, \quad \text{so } y-1 = C(x^2+3)$$

(different const. C)

initial value $y(1)=9$:

$$8 = C\cdot(1^2+3)$$

$$8 = C\cdot 4 \rightarrow \underline{C=2}, \text{ so}$$

$$y-1 = 2\cdot(x^2+3)$$

$$\underline{\underline{y}} = 2x^2+6+1 = \underline{\underline{2x^2+7}}$$

2. The general solution of $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ is:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y} + 3\frac{y}{x}\right)$$ → homogeneous equation

→ substitution $\boxed{v = \frac{y}{x}}$, or $\boxed{xv = y}$

$\rightsquigarrow xv' + v = y'$

$$\rightarrow \quad x\frac{dv}{dx} + v = \frac{1}{2}\cancel{x}\frac{1}{v} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = \frac{1}{2}\cdot\frac{1}{v} + \frac{1}{2}v \quad \text{sep. of variables}$$

$$\frac{2\,dv}{\frac{1}{v}+v} = \frac{dx}{x} \rightarrow \ln(x) = \int \frac{2\,dv}{\frac{1}{v}+v} = \int \frac{2v\,dv}{v^2+1} = \left| \begin{array}{l} t = v^2+1 \\ dt = 2v\,dv \end{array} \right|$$

$$= \int \frac{dt}{t} = \ln(t) + C = \ln(v^2+1) + C$$

$$= \ln\left(\frac{y^2}{x^2}+1\right) + C$$

Exponentiating, we get $x = \left(\frac{y^2}{x^2}+1\right)\cdot c$ (← again, different const. c)

$$c\,x = \frac{y^2}{x^2}+1$$ (← yet again differen const. c)

$$\boxed{y^2+x^2 = Cx^3}$$

$\boxed{3.}$ A tank originally contains 100 gal of water with a salt concentration $\frac{1}{2}$ lb/gal. A solution containing a salt concentration of 2 lb/gal enters at a rate 2 gal/min, and the well-stirred mixture is pumped out at the rate of 1 gal/min. The amount of salt in the tank after 50 min is:

S = overall amount of salt in tank

$S(0) = 50$ lb $\left(= 100 \text{ gal} \cdot \frac{1}{2} \text{lb/gal}\right)$

c = concentration of salt in tank,

$$c(t) = \frac{S}{\underbrace{100+t}_{\text{volume of water in tank}}}$$ ← amount of salt in tank

~> equation $$\frac{dS}{dt} = \underbrace{4}_{\substack{4 \text{ lb/min} \\ \text{of salt going in}}} - \underbrace{\frac{S}{100+t}\cdot 1}_{\substack{(\text{concentration})\cdot 1 \text{ lb} \\ \text{of salt goes out} \\ \text{each min}}}$$

Solve by int. factor: $$\frac{dS}{dt} + \frac{1}{100+t}S = 4$$

$$\mu(t) = e^{\int \frac{1}{100+t}dt} = e^{\ln(100+t)} = 100+t$$

$$S = \frac{\int 4\cdot(100+t)\,dt}{100+t} = \frac{400t+2t^2+C}{100+t}$$

when $t=0$, $S(0)=50$: $50 = \frac{C}{100}$ ~> $\boxed{C = 5000}$

Now when $t=50$, $S(50) = \frac{400\cdot 50 + 2\cdot 2500 + 5000}{150} = 200$ lb

4. Solve the differential eqn $\frac{dy}{dX} - \frac{2}{x} y = x^2 - 1, \quad x > 0$:

Int. factor : $\mu(x) = e^{\int -\frac{2}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}$

$$\rightsquigarrow \underline{\underline{y}} = \frac{\int (x^2-1) \cdot \frac{1}{x^2} dx}{\frac{1}{x^2}} = x^2 \cdot \left(\int (1 - x^{-2}) dx \right) =$$

$$= x^2 \cdot (x + x^{-1} + C) = \underline{\underline{x^3 + x + Cx^2}}$$

5. Find the implicit solution to the initial value problem $(e^x \sin y - 2y \sin x - 1) + (e^x \cos y + 2\cos x + 3) \frac{dy}{dx} = 0, \quad y(0) = \pi$:

$$(\underbrace{e^x \sin y - 2y \sin x - 1}_{M(x,y)}) dx + (\underbrace{e^x \cos y + 2\cos x + 3}_{N(x,y)}) dy = 0$$

$$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x, \quad \frac{\partial N}{\partial x} = e^x \cos y - 2\sin x = \frac{\partial M}{\partial y} \Rightarrow \underline{\text{exact equation}}$$

$\underline{F(x,y) \text{ st } \frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N}$: $F = \int M(x,y) dx = \int (e^x \sin y - 2y \sin x - 1) dx$

$$= e^x \sin y + 2y \cos x - x + g(y),$$

$$e^x \cos y + 2\cos x + 3 = N = \frac{\partial F}{\partial y} = \frac{d}{dy}\left(e^x \sin y + 2y \cos x - x + g(y)\right) =$$

$$= e^x \cos y + 2\cos x + g'(y) \Rightarrow g'(y) = 3$$

$$\rightsquigarrow F = e^x \sin y + 2y \cos x - x + 3y \qquad g(y) = 3y$$

$\rightsquigarrow$ solutions of the form $e^x \sin y + 2y \cos x - x + 3y = C$. Plugging in $\underline{x=0}, \underline{y=\pi}$ yields $C = 5\pi$ $\rightsquigarrow$ $\underline{\underline{e^x \sin y + 2y \cos x - x + 3y = 5\pi}}$

6. Find the general solution of the Bernoulli equation $y' + 2x^{-1}y = 6x^4y^2$:

Substitution $v = y^{-1}$ ⇝ $\frac{dv}{dx} = -y^{-2} \cdot \frac{dy}{dx}$, i.e. $\frac{dy}{dx} = -y^2 \frac{dv}{dx}$

Plugging in, the equation becomes

$$-y^2 \frac{dv}{dx} + 2x^{-1} \underbrace{v y^2}_{(=y)} = 6x^4y^2$$

$$-\frac{dv}{dx} + 2x^{-1}v = 6x^4$$

$$\frac{dv}{dx} - 2x^{-1}v = -6x^4$$

int. factor $\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$

$$v = \frac{\int (-6x^4)\cdot \frac{1}{x^2}dx}{\frac{1}{x^2}} = x^2\left(\int -6x^2 dx\right) = x^2(-2x^3 + C)$$

$$= -2x^5 + Cx^2$$

$$y = v^{-1} = \frac{1}{v} = \frac{1}{-2x^5 + Cx^2}$$

7. Determine all values of k for which the system

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 3x_2 + 2x_3 = 5$$

$$2x_1 + 3x_2 + (k^2-2)x_3 = k+3$$

has no solution.

Matrix of the system

$$\begin{array}{c} \\ (-2x) \\ (-2x) \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 2 & 3 & k^2-2 & k+3 \end{array}\right] \sim \begin{array}{c} \\ \\ (-1x) \end{array}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & k^2-4 & k-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right]$$

This has no solution only if $\underline{k^2 - 4 = 0}$ AND $\underline{k - 2 \neq 0}$

$k^2 - 4 = 0 \iff k = \pm 2$ ... ~~so k=~~ for $k=2$, there is still solution (since $k-2=0$)

⇝ no solution only when $\underline{k = -2}$

8. $A = \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix}$, find all values x, y for which $A \cdot A^T = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$:

$$A \cdot A^T = \begin{bmatrix} 1 & x \\ y & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ x & 2 \end{bmatrix} = \begin{bmatrix} 1+x^2 & y+2x \\ y+2x & y^2+4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

$\rightsquigarrow$ $1 + x^2 = 2$ $\qquad$ $y^2 + 4 = 8$

$x^2 = 1 \ \dots\ x = 1$ or $x = -1$ $\qquad$ $y^2 = 4 \ \dots\ y = 2$ or $y = -2$

Also have to have $\underline{y + 2x = 4}$ out of the four options $\begin{pmatrix} x=-1, y=-2 \\ x=-1, y=2 \\ x=1, y=-2 \\ x=1, y=2 \end{pmatrix}$,

this is satisfied only for the solution $\underline{\underline{x = 1, y = 2}}$

9. The largest open interval on which the solution to the initial value problem $(\cos t)\,y' + \frac{t}{t-4}\,y = \ln(5-t)$, $y(2) = 0$ is guaranteed by the Existence and Uniqueness Thm to exist, is:

$$\left. y' = \frac{\ln(5-t) - \frac{t}{t-4}\,y}{\cos t} \right\} =: f(t,y),$$

$$\frac{\partial f}{\partial y} = -\frac{t}{\cos t\,(t-4)}$$

$f(t,y)$, $\frac{\partial f}{\partial y}$ have to be continuous, in an interval containing $\underline{t=2}$

(in a rectangle containing $(t,y) = (2,0)$)

(*) $f(t,y)$ continuous when
- $\cos t \neq 0 \iff t \notin \{\dots, -\frac{3}{2}\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3}{2}\pi, \dots\}$
- $t \neq 4$
- $5 - t > 0 \iff t < 5$

$\frac{\partial f}{\partial y}$ continuous when $\cos t \neq 0$, $t \neq 4$

maximal interval for t satisfying (*) is $\underline{\underline{\frac{\pi}{2} < t < 4}}$

[$t \neq \frac{\pi}{2}$, $\cos t \neq 0$!] $\frac{\pi}{2}$ ($\cdots$ 2 $\cdots$) 4 [$t \neq 4$!]

$\boxed{10.}$ An object with initial temperature 32°F is placed in a refrigerator whose constant temperature is 0°F. An hour later the temperature of the object is 16°F. What will be its temperature four hours after it is placed in the refrigerator?

Newton's law of cooling: $\frac{dT}{dt} = -k(T - \underbrace{T_m}_{=0})$

$$\rightsquigarrow \frac{dT}{dt} = -kT$$

$$\frac{dT}{T} = -k\,dt$$

$$\int \frac{dT}{T} = \int -k\,dt$$

$$\ln T = -kt + C$$

$$T = C \cdot e^{-kt} \quad \text{(different } C\text{)}$$

- To determine C: $T(0) = 32$

$$32 = C \cdot e^0 = C \quad \rightsquigarrow \quad T = 32 \cdot e^{-kt}$$

- To determine k, or e^{-k}: $T(1) = 16$

$$16 = 32 \cdot e^{-k \cdot 1}$$

$$\boxed{\frac{1}{2} = e^{-k}} \quad \rightsquigarrow \quad T = 32 \cdot \left(\frac{1}{2}\right)^t$$

- after four hours, the temperature is

$$T(4) = 32 \cdot \left(\frac{1}{2}\right)^4 = 2\ {}^\circ F$$

$\boxed{11.}$

For what value of k is the vector $(2,2,1,1)$ in the span of $(1,2,1,-1)$ and $(3,2,1,k)$?

write vectors in columns instead (does not change anything)

$\rightsquigarrow$ The question is when does the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \\ 1 \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

have a solution.

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \\ -1 & k & 1 \end{array}\right] \begin{array}{l} \\ (-2)\times \\ (-1)\times \\ (+1)\times \end{array} \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -4 & -2 \\ 0 & -2 & -1 \\ 0 & k+3 & 3 \end{array}\right] \begin{array}{l} \\ \cdot\left(-\frac{1}{2}\right) \\ \cdot(-1) \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & k+3 & 3 \end{array}\right] (-1)\times \sim$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & k+3 & 3 \end{array}\right]$$

from here, we see that

$\underline{x_2 = \frac{1}{2}}$, $\underline{x_1 = \frac{1}{2}}$

so for consistency of the system, one has to have

$$\underbrace{(k+3) \cdot \frac{1}{2}}_{x_2} = 3$$

$$k+3 = 6$$

$$\boxed{k = 3}$$