1.

A $n \times n$ nonsingular matrix. Then which of these are true?

(i) $\det A = 0$ **FALSE** ($\det A = 0 \iff A$ is singular)

(ii) $\operatorname{rank} A = n$ **TRUE**

(iii) $A\vec{x} = 0$ has infinitely many solutions **FALSE** (A nonsingular $\Rightarrow \operatorname{rank} A = n \Rightarrow$ the only solution to $A\vec{x} = 0$ is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$)

(iv) $A\vec{x} = \vec{b}$ has a unique solution for every vector $\vec{b} \in \mathbb{R}^n$

TRUE (the unique solution is $\vec{x} = A^{-1}\vec{b}$)

(v) A is row equivalent to I_n ... $n \times n$ identity matrix

TRUE

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2. Consider the i.v.p. $t(t-10)y'' + y' - \frac{1}{t-3}y = \ln(t-5)$,

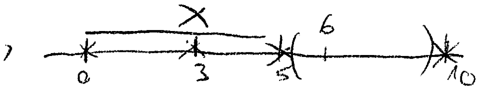
$$y(6) = 0, \quad y'(6) = 1$$

Find the largest interval on which the i.v.p. has a unique solution.

~~Existence and~~ Rewrite the equation to $y'' + \frac{1}{t(t-10)}y' - \frac{1}{(t-3)t(t-10)}y = \frac{\ln(t-5)}{t(t-10)}$

Existence & uniqueness theorem $\Rightarrow$ need all the ~~three~~ functions to be continuous ~~about~~ in an interval containing $t = 6$ (since the initial conditions talk about $y(6)$, $y'(6)$)

$\frac{1}{t(t-10)}$ continuous when $t \neq 0, 10$, $\frac{1}{t(t-3)(t-10)}$ cts when $t \neq 0, 3, 10$; $\frac{\ln(t-5)}{t(t-10)}$ cts when $t > 5$, $t \neq 0$, $t \neq 10$

$\rightsquigarrow$ 

biggest such interval is $(5, 10)$.

3.

Which of the following subsets $S \subseteq V$ is a subspace of V?

(i) $V = \mathbb{R}^3$, S = all vectors (x,y,z) satisfying $x + 2y - 3z = 0$

... it is a subspace

$0 + 2 \cdot 0 - 3 \cdot 0 = 0 \rightarrow (0,0,0) \in S$

$(x_1,y_1,z_1), (x_2,y_2,z_2) \in S \Rightarrow$ $x_1 + 2y_1 - 3z_1 = 0$ (A)

$x_2 + 2y_2 - 3z_2 = 0$ (B)

add (A),(B) $\rightsquigarrow$ $(x_1+x_2) + 2(y_1+y_2) - 3(z_1+z_2) = 0$

$\Rightarrow (x_1+x_2, y_1+y_2, z_1+z_2) \in S$.

similarly $(x,y,z) \in S$, c scalar $\Rightarrow (cx, cy, cz) \in S$

(OR: Any set of all solutions to a system of linear homogeneous equations forms a subspace)

(ii) $V = M_2(\mathbb{R})$ (= 2×2 real matrices), S = all 2×2 matrices with determinant $\neq 0$

... it is not a subspace (the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not belong to S)

(iii) $V = P_2$ (= polynomials of degree ≤ 2), S = all the polynomials of the form $ax^2 - bx$, $a, b \in \mathbb{R}$

... it is a subspace ($S = \text{Span}\{x, x^2\}$ and linear span of any set is always a subspace)

(iv) $V = M_n(\mathbb{R})$ (= $n\times n$ real matrices), S = the set of all non-symmetric matrices ... it is not a subspace (the zero matrix $\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$ is symmetric, hence not in S)

4. Determine the general solution to $(D+1)(D-1)^2(D^2+2D+2)\,y=0$: (*)

$(D+1)$ annihilates $y_1 = e^{-x}$

$(D-1)^2$ annihilates $y_2 = e^{x}$, $y_3 = xe^{x}$

(D^2+2D+2) annihilates ...?

$(D^2+2D+2)y=0 \iff y''+2y'+2y=0$, char eqn $r^2+2r+2=0$

roots $r_{1,2} = \dfrac{-2\pm\sqrt{4-8}}{2} = -1\pm i$

$\leadsto$ solutions $y_4 = e^{-x}\cos x$, $y_5 = e^{-x}\sin x$

Altogether: The general solution to (*) is

$$y = c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 + c_5y_5 =$$
$$= c_1e^{-x} + c_2e^{x} + c_3xe^{x} + e^{-x}(c_4\cos x + c_5\sin x)$$

5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. What is a basis of $Col(A)$?

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} (-1)\times \\ (-2)\times \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & -6 & -6 & -12 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \text{divide by } (-2) \\ \text{divide by } (-6) \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ (-1)\times \\ (-1)\times \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

$\leadsto$ corresponding columns of A form the basis of $Col(A)$, i.e.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis of $Col(A)$.

[6.] The general solution to $y'''' - 8y'' + 16y = 0$ is:

Aux eqn $r^4 - 8r^2 + 16 = 0$. Let $t = r^2 \leadsto t^2 - 8t + 16 = 0$

$$t_{1,2} = \frac{8 \pm \sqrt{8^2 - 4\cdot 16}}{2} = 4 \pm 0 = 4$$

$\Rightarrow t^2 - 8t + 16 = (t-4)^2$

$\Rightarrow r^4 - 8r^2 + 16 = (r^2 - 4)^2 = ((r-2)\cdot(r+2))^2 = (r-2)^2 \cdot (r+2)^2$

$\rightarrow$ roots $r_{1,2} = 2$, $r_{3,4} = -2$

$\leadsto$ general sol. is of the form $y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$

[7.] $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, what is a basis of Nul(A)?

$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ solutions of the form $\left\{ \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} \;\middle|\; r, s \in \mathbb{R} \right\} =$

pivot, free vars

$= \left\{ r \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; r, s \in \mathbb{R} \right\}$

$\Rightarrow$ a basis of Nul(A) is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

[8.] Let $y(t)$ be the solution to the ivp. $y'' + 3y' - 4y = 6e^{2t}$, $y(0) = 2$, $y'(0) = 3$.

Find $y(1)$:

y_h: $y'' + 3y' - 4y = 0$, aux eqn $r^2 + 3r - 4 = 0$

$(r+4)(r-1) = 0$ $\quad r_1 = 1$, $r_2 = -4$

$\leadsto y_h = c_1 e^{t} + c_2 e^{-4t}$

y_p: undetermined coeff, looking for $y_p = Ae^{2t}$ $\quad y_p' = 2Ae^{2t}$

$y_p'' = 4Ae^{2t}$

→ plug in: $4Ae^{2t} + 3\cdot(2Ae^{2t}) - 4Ae^{2t} = 6e^{2t}$

$6Ae^{2t} = 6e^{2t} \Rightarrow A=1$, so $y_p = e^{2t}$

→ gen. sol. $y = y_h + y_p = c_1 e^{t} + c_2 e^{-4t} + e^{2t}$

initial conditions:

$$2 = y(0) = c_1 + c_2 + 1$$
$$3 = y'(0) = c_1 - 4c_2 + 2$$

$$\begin{cases} c_1 + c_2 = 1 & (A) \\ c_1 - 4c_2 = 1 & (B) \end{cases}$$

(A) − (B): $5c_2 = 0 \rightarrow c_2 = 0$, so $c_1 = 1 \Rightarrow y = e^{t} + e^{2t}$,

so $y(1) = e + e^2$

9. Given that $y_1(t) = t$ is a solution to $t^2y'' - ty' + y = 0$, $t > 0$, find a second linearly independent solution $y_2(t)$:

Method 1 (more general, reduction of order):

Look for $y_2(t) = v\cdot y_1(t) = v\cdot t$, v a function. $\left(\begin{array}{l} y_2' = v't + v \\ y_2'' = v''t + 2v' \end{array} \right)$

Plug in:

$$t^2(v''t + 2v') - t(v't + v) + vt = 0$$
$$v''t^3 + v't^2 = 0$$
$$v''t + v' = 0 \qquad \text{set } u = v'$$
$$\rightsquigarrow \quad u't + u = 0 \qquad \text{sep. equation}$$
$$u't = -u$$
$$\frac{u'}{u} = -\frac{1}{t}$$

$\ln(u) = \int \frac{u'}{u}\,dt = \int -\frac{1}{t}\,dt = -\ln t$

$v' = u = e^{-\ln t} = \frac{1}{t}$

$v = \int \frac{1}{t}\,dt = \ln(t)$, so

$y_2(t) = v\cdot t = \ln(t)\cdot t$

Method 2 (specific to Cauchy-Euler equations)

Solve the C.-E. equation $t^2y''-ty'+y=0$, $t>0$ directly:

Characteristic eqn $r(r-1)-r+1=0$

$$r^2-2r+1=0$$

$$(r-1)^2=0 \rightsquigarrow \text{roots } r_{1,2}=1 \quad \text{(double root)}$$

$\Rightarrow$ lin. indep. solutions are $y_1=t$ and $y_2=\ln(t)\cdot t$

$\boxed{10.}$ Let $\lambda=3$ be an eigenvalue of $A=\begin{bmatrix}1&2&2\\-1&4&1\\0&0&3\end{bmatrix}$. Then the geometric multiplicity of $\lambda=3$ is:

geom. multiplicity of $\lambda=3$ = dimension of the ~~eig~~ eigenspace for $\lambda=3$, ie of ~~space of~~ the nullspace of the matrix

$$A-3\cdot I=\begin{bmatrix}-2&2&2\\-1&1&1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-1&-1\\0&0&0\\0&0&0\end{bmatrix}$$

pivot free

2 free variables

$\Rightarrow$ dim $=2$

$\Rightarrow$ geom. multiplicity of $\lambda=3$ is 2

$\boxed{11.}$ Find the solution to the i.v.p. $y''+y=4\sin(t)$, $y(0)=2$, $y'(0)=0$

Aux eqn $r^2+1=0 \Rightarrow r_{1,2}=\pm i \Rightarrow y_h=c_1\cos t+c_2\sin t$

y_p of the form $y_p=At\cos t+Bt\sin t$ (t-multiple since $\pm i$ are roots, so $A\cos t+B\sin t$ solve the homogeneous equation.)

After plugging in one gets $-2A\sin t+2B\cos t=4\sin t \Rightarrow B=0,\ A=-2$

so $y_p=-2t\cos t$, $y=c_1\cos t+c_2\sin t-2t\cos t$

initial conditions: $2=y(0)=c_1$ $\quad(y'=-c_1\sin t+(c_2-2)\cos t+2t\sin t)$

$0=y'(0)=c_2-2$ $\Big\}\Rightarrow y=2\cos t+2\sin t-2t\cos t$