Quiver Algebras

Pavel Čoupek

Purdue University

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Motivation: ”Matrix problems”

Fix an algebraically closed field $k$.

**Problem**

Classify all members of a set $S$ of $(n$-tuples of) matrices over $k$ up to a ”linear-algebraic” equivalence $\sim$. Find representatives of the equivalence classes.
Motivation: "Matrix problems"

Example

$S = \text{all matrices, } A \sim B \text{ iff } A = SBT^{-1}$ for some regular square matrices $S, T$

Solution: $A \sim B$ if and only if they are of the same dimensions and $\text{rank}A = \text{rank}B$. Canonical representatives are

$$
\begin{pmatrix}
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$
Motivation: ”Matrix problems”

Example

$S = \text{all square matrices, } A \sim B \text{ iff } A = SBS^{-1} \text{ for some regular square matrix } S$

Solution: $A \sim B$ if and only if they have the same structure of generalized eigenspaces. Canonical representatives are

$$
\begin{pmatrix}
J_1 \\
& J_2 \\
& & \ddots \\
& & & J_k
\end{pmatrix},
$$

where $J_i$'s are the Jordan blocks.
Two subspace problem

Example

$S =$ pairs of matrices $(A_1, A_2)$ with the same number of rows,

$$(A_1, A_2) \sim (B_1, B_2) \iff B_1 = S A_1 T_1^{-1}, B_2 = S A_2 T_2^{-1}$$

for some regular matrices $S, T_1, T_2$

Subspace form:

Given a $k$-vector space $V$ and $V_1, W_1, V_2, W_2 \leq V$ its subspaces, when does an automorphism $f : V \simrightarrow V$ exist such that $f(V_1) = W_1$ and $f(V_2) = W_2$?
$n$—subspace problem

Example

$S = n$-tuples of matrices $(A_1, A_2, \ldots, A_n)$ with the same number of rows,

$$(A_1, \ldots, A_n) \sim (B_1, \ldots, B_n) \quad \iff \quad B_i = SA_iT_1^{-1}, \ i = 1, 2, \ldots, n,$$

for some regular matrices $S, T_1, T_2, \ldots, T_n$

Subspace form:

Given a $k$-vector space $V$ and $V_i, W_i \leq V, \ i = 1, \ldots, n$, its subspaces, when does an automorphism $f : V \xrightarrow{\sim} V$ exist such that $f(V_i) = W_i, \ i = 1, \ldots, n$?
Kronecker problems

Example (Kronecker problem)

\[ S = \text{pairs of matrices } (A_1, A_2) \text{ of the same dimensions,} \]

\[(A_1, A_2) \sim (B_1, B_2) \iff B_i = S A_i T^{-1}, \ i = 1, 2\]

Example (3-Kronecker problem)

\[ S = \text{triples of matrices } (A_1, A_2, A_3) \text{ of the same dimensions,} \]

\[(A_1, A_2, A_3) \sim (B_1, B_2, B_3) \iff B_i = S A_i T^{-1}, \ i = 1, 2, 3\]
Simultaneous similarity

Example (pairs of sim. similar matrices)

$S =$ pairs of square matrices $(A_1, A_2)$ of the same order,

$$(A_1, A_2) \sim (B_1, B_2) \iff B_i = S A_i S^{-1}, \ i = 1, 2$$

Example ($n-$tuples of sim. similar matrices)

$S =$ triples of square matrices $(A_1, A_2, \ldots A_n)$ of the same order,

$$(A_1, A_2, \ldots, A_n) \sim (B_1, B_2, \ldots, B_n) \iff B_i = S A_i S^{-1}, \ \forall i$$
Example (2-subspace problem)

\[ B_1 = SA_1 T_1^{-1}, B_2 = SA_2 T_2^{-1} \]

is, coordinate-freely, the commutativity of

\[
\begin{array}{c}
V_1 \xrightarrow{A_1} V \xleftarrow{A_2} V_2 \\
\sim \downarrow T_1 \sim \downarrow S \sim \downarrow T_2 \\
W_1 \xrightarrow{B_1} W \xleftarrow{B_2} W_2
\end{array}
\]
Example (Kronecker problem)

\[ B_1 = SA_1T^{-1}, B_2 = SA_2T^{-1} \]

translates to the commutativity of

\[ \begin{array}{cc}
  V_1 & V_2 \\
  \downarrow T & \downarrow S \\
  W_1 & W_2 \\
\end{array} \]

\[ \begin{array}{cc}
  \overset{A_2}{\longrightarrow} & \overset{A_1}{\longrightarrow} \\
  \underset{B_2}{\sim} & \underset{B_1}{\sim} \\
\end{array} \]

(in the respective squares).
Representations of quivers

Definition
A *quiver* $Q = (V, E)$ consists of

- a finite collection of vertices $V$,
- a finite set of oriented edges $E$ between them;

multiple edges and loops are allowed.

Denote $s : E \rightarrow V$, $t : E \rightarrow V$ the source and target functions.

Definition
A *$k$-linear representation of a quiver* $Q = (V, E)$ consists of

- for each vertex $v$, a $k$-linear space $M_v$,
- for each edge $\alpha$, a $k$-linear map $f_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$.

A *homomorphism of rep’s* $(M_v, f_\alpha) \rightarrow (M'_v, f'_\alpha)$ is a collection of $k$-linear maps $g_v : M_v \rightarrow M'_v$ compatible with the edge maps.
Example (2-subspace)

The datum of vector spaces and linear maps

$$V_1 \xrightarrow{A_1} V \xleftarrow{A_2} V_2$$

is a representation of the quiver

$$\begin{array}{ccc}
1 & \xrightarrow{\alpha} & 3 \\
\bullet & & \bullet \\
\end{array} \quad \quad \begin{array}{ccc}
3 & \xleftarrow{\beta} & 2 \\
\bullet & & \bullet \\
\end{array}$$

Example (Kronecker)

The datum of vector spaces and linear maps

$$V_1 \xrightarrow{A_2} V_2 \quad \xleftarrow{A_1} V_1$$

is a representation of the quiver

$$\begin{array}{ccc}
1 & \xrightarrow{\beta} & 2 \\
\bullet & & \bullet \\
\end{array} \quad \quad \begin{array}{ccc}
2 & \xleftarrow{\alpha} & 1 \\
\bullet & & \bullet \\
\end{array}$$
The corresponding quivers to other matrix problems are:

- **rank problem**

- **similarity problem**

- **3-subspace problem**

- **2-similarity problem**
Path algebra of a quiver

**Definition**
A *path* in a quiver $Q$ is either a vertex, or sequence of arrows $p = \alpha_n\alpha_{n-1} \cdots \alpha_1$ such that $s(\alpha_i) = t(\alpha_{i-1})$.

**Definition**
A *path algebra* $kQ$ of a quiver $Q = (V, E)$ is given by
- the vector space with basis = the set of all paths,
- multiplication given on the basis by

$$p \cdot q = \begin{cases} 
\text{the path } pq \text{ if they connect}, \\
0 \text{ otherwise.} 
\end{cases}$$
Proposition (and proof)

To give a representation \((M_v, f_\alpha)\) of quiver \(Q\) is to give a left \(kQ\)-module \(M\):

\[
M \overset{\text{def}}{=} \bigoplus_v M_v \quad \text{with} \quad \alpha \cdot (m_{v_1} + m_{v_2} + \ldots m_{v_l}) \overset{\text{def}}{=} f_\alpha(m_{s(\alpha)}).
\]

The representation of \(Q\) can be recovered from \(M\) by setting \(M_v = v \cdot M\) and \(f_\alpha = [\alpha \cdot -]\).

Thus, to solve a given matrix problem is equivalent to classification of finite-dimensional modules of the corresponding quiver algebra.
Some examples

1
\[ \begin{array}{cc}
\bullet & P \\
\bullet & \rightarrow \\
\bullet & \rightarrow \bullet
\end{array} \]

\[ k[X] \]

1
\[ \begin{array}{cc}
\bullet & \rightarrow \\
\bullet & \rightarrow \bullet
\end{array} \]

\[ \begin{pmatrix}
k & k \\
0 & k
\end{pmatrix} \]

1
\[ \begin{array}{cc}
\bullet & \rightarrow \\
\bullet & \rightarrow \bullet
\end{array} \]

... \rightarrow \bullet

Upper triangular \( n \times n \) matrices

1
\[ \begin{array}{cc}
\bullet & \rightarrow \\
\bullet & \leftrightarrow \\
\bullet & \rightarrow \\
\bullet & \rightarrow \bullet
\end{array} \]

\[ k\langle X, Y \rangle \text{ (2 noncomm. free var's)} \]
Indecomposable modules and representation types

Let $A$ be a $k$-algebra and $M$ a left $A$-module of finite dimension.

**Definition**

$M$ is called *indecomposable* if $M \neq 0$ and whenever $M = M_1 \oplus M_2$, then $M_1 = M$ and $M_2 = 0$, or vice versa.

**Theorem (Krull-Schmidt-Remak-Azumaya)**

Given a fin.-dim. left $A$-module $M$, it can be written as $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$'s are indecomposable, and this decomposition is unique (up to permutation and isomorphism of the factors).

*Our goal revised:* Classify all the indecomposable fin.-dim. modules of a given quiver algebra.
Trichotomy of representation type

Definition (not really a definition)

A $k$-algebra $A$ is

- **of finite representation type** if there are only finitely many indecomposable fin.-dim. $A$-modules (e.g. the rank problem)
- **of tame representation type** if the indecomposable modules can form countably many one-parameter families $M_{\lambda}$, $\lambda \in k$ (e.g. the similarity problem)
- **of wild representation type** otherwise. (e.g. the 2-similarity problem)

Theorem (Drozd; "wild type is bad")

Given an algebra $A$ of wild representation type and $\Lambda$ any fin.-dim. algebra, there is an exact functor $\Lambda-\text{mod} \to A-\text{mod}$ preserving (and not identifying) indecomposables.
Theorem (Gabriel)

A quiver $Q$ is of finite representation type if and only if its underlying non-oriented graph is a disjoint union of the Dynkin diagrams $A_n, D_n, E_6, E_7, E_8$:

$A_n$

1 \[ \bullet \quad 2 \quad 3 \quad \ldots \quad n-1 \quad n \]

$D_n$

1 \[ \bullet \quad 2 \quad 3 \quad \ldots \quad n-2 \quad n-1 \]

$\quad | \quad n \quad \bullet$
Theorem (Gabriel)

A quiver $Q$ is of finite representation type if and only if its underlying non-oriented graph is a disjoint union of the Dynkin diagrams $A_n, D_n, E_6, E_7, E_8$:
Theorem (Nazarova)

A quiver $Q$ is of tame representation type if and only if its underlying graph is a disjoint union of $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, and the extended Dynkin diagrams $\hat{A}_n$, $\hat{D}_n$, $\hat{E}_6$, $\hat{E}_7$, $\hat{E}_8$:
Theorem (Nazarova)

A quiver $Q$ is of tame representation type if and only if its underlying graph is a disjoint union of $A_n, D_n, E_6, E_7, E_8$, and the extended Dynkin diagrams $\widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8$: 

1. $\widehat{E}_6$

2. $\widehat{E}_7$

3. $E_8$
The 3-subspace problem is of finite representation type \((D_4)\); the indecomposables are (up to ”permutation of legs”):

\[
\begin{align*}
0 & \xrightarrow{0} k \leftarrow 0 & 0 \\
\uparrow & & \uparrow \\
0 & & 0
\end{align*}
\begin{align*}
k & \xrightarrow{0} 0 \leftarrow 0 & k \\
\uparrow & & \uparrow \\
0 & & 0
\end{align*}
\begin{align*}
k & \xrightarrow{1} k & k \leftarrow 0 \\
\uparrow & & \uparrow \\
k & & k
\end{align*}
\begin{align*}
k & \xrightarrow{(1)} k \oplus k & k \leftarrow (0) \\
(1) & & (1) \\
k & & k
\end{align*}
The 4-subspace problem is not of finite representation type; there is a family of indecomposables indexed by $\lambda \in k$, $\lambda \neq 0, 1$:

$$
\begin{align*}
\begin{array}{c}
k \\
\downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
k \oplus k \oplus k \oplus k \\
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{array}
\end{align*}
$$

It is, however, of tame representation type ($\widehat{D}_4$): Nothing worse than the above can happen.
The 5-subspace problem is of wild representation type; there is a family of indecomposables indexed by $\lambda, \sigma \in k$:
General (bounded) quiver algebras

Given a path algebra of a quiver $kQ$, denote by $R_Q$ the two-sided ideal generated by all arrows of $Q$.

**Definition**

A two-sided ideal ideal $\mathfrak{a} \subseteq kQ$ is *admissible* if $R_Q^m \subseteq \mathfrak{a} \subseteq R_Q^2$ for some $m \geq 0$. In that case we call $A = kQ/\mathfrak{a}$ a quiver algebra (of quiver $Q$ with relations $\mathfrak{a}$).

**Example**

$$
\begin{align*}
Q : & \quad 1 \rightarrow^\alpha 2 \\
& \quad \downarrow^\gamma \quad \downarrow^\beta \\
& \quad 3 \rightarrow^\delta 4
\end{align*}
$$

$\mathfrak{a} = \langle \beta\alpha - \delta\gamma \rangle$

Then $(kQ/\mathfrak{a})$-modules are precisely the representations of $Q$ for which the square is commutative.
Definition
Rings $R, S$ are Morita equivalent if $R$-Mod and $S$-Mod are equivalent as additive categories.
(If $R, S$ are fin.-dim. algebras, the same is true for the full subcategories of fin.-dim. modules.)

Theorem (Gabriel)
Let $A$ be any finite-dimensional algebra over an algebraically closed field. Then $A$ is Morita equivalent to a bounded quiver algebra (of a finite quiver).

In particular, to provide a classification of bounded quiver algebras to finite/tame/wild type amounts to classifying all finite-dimensional algebras.