Quiver Algebras

Pavel Čoupek

Purdue University

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Motivation: "Matrix problems"

Fix an algebraically closed field $\mathbf{k}$.

Problem

Classify all members of a set $\mathcal{S}$ of (n-tuples of) matrices over $\mathbf{k}$ up to a "linear-algebraic" equivalence $\sim$.
Find representatives of the equivalence classes.

Motivation: "Matrix problems"

Example

$\mathcal{S}$ = all matrices, $A \sim B$ iff $A = SBT^{-1}$ for some regular square matrices S, T

Solution: $A \sim B$ if and only if they are of the same dimensions and $\mathrm{rank} A = \mathrm{rank} B$. Canonical representatives are

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Motivation: "Matrix problems"

Example

$\mathcal{S}$ = all square matrices, $A \sim B$ iff $A = SBS^{-1}$ for some regular square matrix S

Solution: $A \sim B$ if and only if they have the same structure of generalized eigenspaces. Canonical representatives are

$$\begin{pmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{pmatrix},$$

where $\boxed{J_i}$'s are the Jordan blocks.

Two subspace problem

Example

$\mathcal{S}$ = pairs of matrices (A_1, A_2) with the same number of rows,

$$(A_1, A_2) \sim (B_1, B_2) \quad \overset{def}{\Leftrightarrow} \quad B_1 = SA_1T_1^{-1}, B_2 = SA_2T_2^{-1}$$

for some regular matrices S, T_1, T_2

Subspace form:

Given a $\mathbf{k}$-vector space V and $V_1, W_1, V_2, W_2 \leq V$ its subspaces, when does an automorphism $f : V \xrightarrow{\simeq} V$ exist such that $f(V_1) = W_1$ and $f(V_2) = W_2$?

$n-$subspace problem

Example

$\mathcal{S} = n$-tuples of matrices $(A_1, A_2, \ldots, A_n)$ with the same number of rows,

$$(A_1, \ldots, A_n) \sim (B_1, \ldots, B_n) \quad \overset{def}{\Leftrightarrow} \quad B_i = SA_iT_1^{-1},\ i = 1, 2, \ldots, n,$$

for some regular matrices $S, T_1, T_2, \ldots, T_n$

Subspace form:

Given a $\mathbf{k}$-vector space V and $V_i, W_i \leq V,\ i = 1, \ldots, n$, its subspaces, when does an automorphism $f : V \xrightarrow{\simeq} V$ exist such that $f(V_i) = W_i,\ i = 1, \ldots, n$?

Kronecker problems

Example (Kronecker problem)

$\mathcal{S} =$ pairs of matrices (A_1, A_2) of the same dimensions,

$$(A_1, A_2) \sim (B_1, B_2) \quad \overset{def}{\Leftrightarrow} \quad B_i = SA_iT^{-1},\ i = 1, 2$$

Example (3-Kronecker problem)

$\mathcal{S} =$ triples of matrices (A_1, A_2, A_3) of the same dimensions,

$$(A_1, A_2, A_3) \sim (B_1, B_2, B_3) \quad \overset{def}{\Leftrightarrow} \quad B_i = SA_iT^{-1},\ i = 1, 2, 3$$

Simultaneous similarity

Example (pairs of sim. similar matrices)

$\mathcal{S} =$ pairs of square matrices (A_1, A_2) of the same order,

$$(A_1, A_2) \sim (B_1, B_2) \quad \overset{def}{\Leftrightarrow} \quad B_i = SA_iS^{-1},\ i = 1, 2$$

Example ($n-$tuples of sim. similar matrices)

$\mathcal{S} =$ triples of square matrices $(A_1, A_2, \dots A_n)$ of the same order,

$$(A_1, A_2, \dots, A_n) \sim (B_1, B_2, \dots, B_n) \quad \overset{def}{\Leftrightarrow} \quad B_i = SA_iS^{-1},\ \forall i$$

Matrix problems in pictures

Example (2-subspace problem)

$$B_1 = SA_1T_1^{-1}, B_2 = SA_2T_2^{-1}$$

is, coordinate-freely, the commutativity of

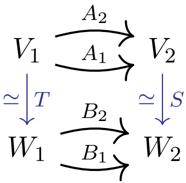
$$\begin{array}{ccccc} V_1 & \xrightarrow{A_1} & V & \xleftarrow{A_2} & V_2 \\ \simeq \downarrow T_1 & & \simeq \downarrow S & & \simeq \downarrow T_2 \\ W_1 & \xrightarrow{B_1} & W & \xleftarrow{B_2} & W_2 \end{array}$$

Matrix problems in pictures

Example (Kronecker problem)

$$B_1 = SA_1T^{-1}, B_2 = SA_2T^{-1}$$

translates to the commutativity of



(in the respective squares).

Representations of quivers

Definition

A *quiver* $Q = (V, E)$ consists of

- a finite collection of vertices V,
- a finite set of oriented edges E between them;

multiple edges and loops are allowed.
Denote $s : E \to V$, $t : E \to V$ the source and target functions.

Definition

A $\mathbf{k}$*-linear representation of a quiver* $Q = (V, E)$ consists of

- for each vertex v, a $\mathbf{k}$-linear space M_v,
- for each edge α, a $\mathbf{k}$-linear map $f_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$.

A *homomorphism of rep's* $(M_v, f_\alpha) \to (M'_v, f'_\alpha)$ is a collection of $\mathbf{k}$-linear maps $g_v : M_v \to M'_v$ compatible with the edge maps.

Example (2-subspace)

The datum of vector spaces and linear maps

$$V_1 \xrightarrow{A_1} V \xleftarrow{A_2} V_2$$

is a representation of the quiver

$$\overset{1}{\bullet} \xrightarrow{\alpha} \overset{3}{\bullet} \xleftarrow{\beta} \overset{2}{\bullet}\ .$$

Example (Kronecker)

The datum of vector spaces and linear maps

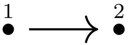
$$V_1 \underset{A_1}{\overset{A_2}{\rightrightarrows}} V_2$$

is a representation of the quiver

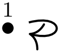
$$\overset{1}{\bullet} \underset{\alpha}{\overset{\beta}{\rightrightarrows}} \overset{2}{\bullet}\ .$$

The corresponding quivers to other matrix problems are:

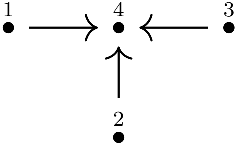
rank problem



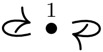
similarity problem



3-subspace problem



2-similarity problem



Path algebra of a quiver

Definition

A *path* in a quiver Q is either a vertex, or sequence of arrows $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ such that $s(\alpha_i) = t(\alpha_{i-1})$

Definition

A *path algebra* $\mathbf{k}Q$ of a quiver $Q = (V, E)$ is given by

- the vector space with basis $=$ the set of all paths,
- multiplication given on the basis by

$$p \cdot q = \begin{cases} \text{the path } pq \text{ if they connect,} \\ 0 \text{ otherwise.} \end{cases}$$

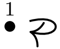
Proposition (and proof)

To give a representation (M_v, f_α) *of quiver* Q *is to give a left* $\mathbf{k}Q$*-module* M*:*

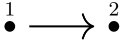
- $M \stackrel{def}{=} \bigoplus_v M_v$ *with* $\alpha \cdot (m_{v_1} + m_{v_2} + \dots m_{v_l}) \stackrel{def}{=} f_\alpha(m_{s(\alpha)})$.
- *The representation of* Q *can be recovered from* M *by setting* $M_v = v \cdot M$ *and* $f_\alpha = [\alpha \cdot -]$.

Thus, to solve a given matrix problem is equivalent to classification of finite-dimensional modules of the corresponding quiver algebra.

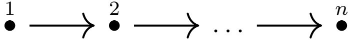
Some examples



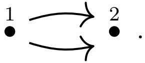
$\mathbf{k}[X]$



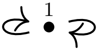
$\begin{pmatrix} \mathbf{k} & \mathbf{k} \\ 0 & \mathbf{k} \end{pmatrix}$



Upper triangular $n \times n$ matrices



$\begin{pmatrix} \mathbf{k} & \mathbf{k} \oplus \mathbf{k} \\ 0 & \mathbf{k} \end{pmatrix}$



$\mathbf{k}\langle X, Y \rangle$ (2 noncomm. free var's)

Indecomposable modules and representation types

Let A be a $\mathbf{k}$-algebra and M a left A-module of finite dimension.

Definition

M is called *indecomposable* if $M \neq 0$ and whenever $M = M_1 \oplus M_2$, then $M_1 = M$ and $M_2 = 0$, or vice versa.

Theorem (Krull-Schmidt-Remak-Azumaya)

Given a fin.-dim. left A-module M, it can be written as $M = \bigoplus_{i=1}^{n} M_i$, where M_i's are indecomposable, and this decomposition is unique (up to permutation and isomorphism of the factors).

Our goal revised: Classify all the indecomposable fin.-dim. modules of a given quiver algebra.

Trichotomy of representation type

Definition (not really a definition)

A $\mathbf{k}$-algebra A is

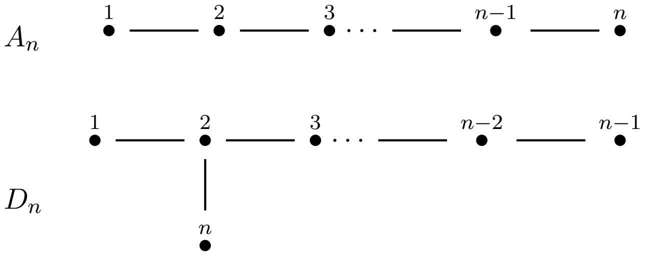
- *of finite representation type* if there are only finitely many indecomposable fin.-dim. A-modules (e.g. the rank problem)
- *of tame representation type* if the indecomposable modules can form countably many one-parameter families M_λ, $\lambda \in \mathbf{k}$ (e.g. the similarity problem)
- *of wild representation type* otherwise. (e.g. the 2-similarity problem)

Theorem (Drozd; "wild type is bad")

Given an algebra A of wild representation type and Λ any fin.-dim. algebra, there is an exact functor $\Lambda-\mathrm{mod} \to A-\mathrm{mod}$ preserving (and not identifying) indecomposables.

Theorem (Gabriel)

A quiver Q is of finite representation type if and only if its underlying non-oriented graph is a disjoint union of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8:



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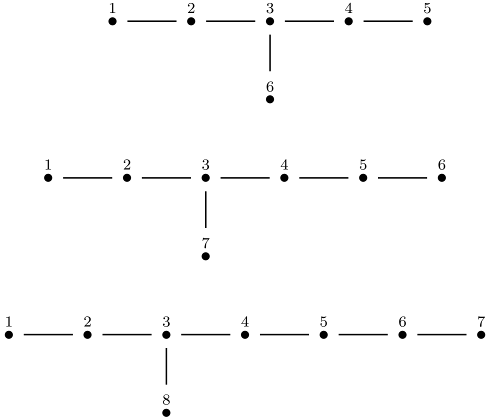
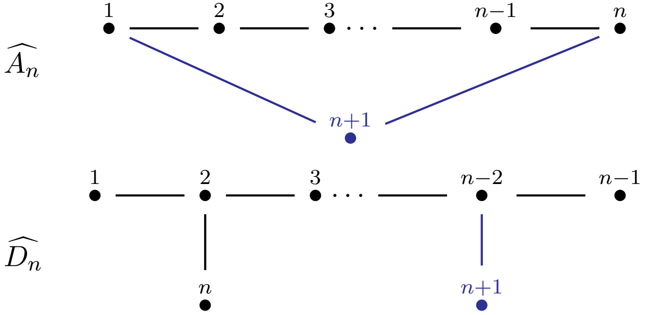
E_6 E_7 E_8



Theorem (Nazarova)

A quiver Q is of tame representation type if and only if its underlying graph is a disjoint union of A_n, D_n, E_6, E_7, E_8, and the extended Dynkin diagrams $\widehat{A_n}, \widehat{D_n}, \widehat{E_6}, \widehat{E_7}, \widehat{E_8}$:



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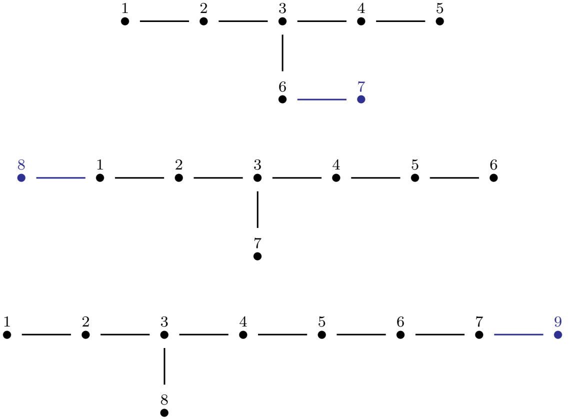
$\widehat{E_6}$ $\widehat{E_7}$ E_8



Back to: n-subspace problem

The 3-subspace problem is of finite representation type (D_4); the indecomposables are (up to "permutation of legs"):

$$\begin{array}{ccccc} 0 & \xrightarrow{0} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{0} & 0 & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{1} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 0}\uparrow & & \\ & & 0 & & \end{array}$$

$$\begin{array}{ccccc} \mathbf{k} & \xrightarrow{1} & \mathbf{k} & \xleftarrow{0} & 0 \\ & & {\scriptstyle 1}\uparrow & & \\ & & \mathbf{k} & & \end{array} \qquad \begin{array}{ccccc} \mathbf{k} & \xrightarrow{\binom{1}{0}} & \mathbf{k} \oplus \mathbf{k} & \xleftarrow{\binom{0}{1}} & \mathbf{k} \\ & & {\scriptstyle \binom{1}{1}}\uparrow & & \\ & & \mathbf{k} & & \end{array}$$

Back to: n-subspace problem

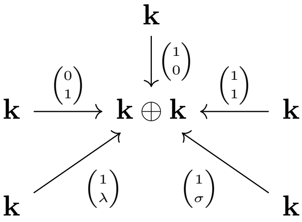
The 4-subspace problem is not of finite representation type; there is a family of indecomposables indexed by $\lambda \in \mathbf{k}, \lambda \neq 0, 1$:

$$
\begin{array}{ccccc}
 & & \mathbf{k} & & \\
 & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & & \\
\mathbf{k} & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & \mathbf{k} \oplus \mathbf{k} & \xleftarrow{\begin{pmatrix} 1 \\ \lambda \end{pmatrix}} & \mathbf{k} \\
 & & {\scriptstyle \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \Big\uparrow & & \\
 & & \mathbf{k} & &
\end{array}
$$

It is, however, of tame representation type $(\widehat{D_4})$: Nothing worse than the above can happen.

Back to: n-subspace problem

The 5-subspace problem is of wild representation type; there is a family of indecomposables indexed by $\lambda, \sigma \in \mathbf{k}$:



General (bounded) quiver algebras

Given a path algebra of a quiver $\mathbf{k}Q$, denote by $\mathfrak{R}_Q$ the two-sided ideal generated by all arrows of Q.

Definition

A two-sided ideal ideal $\mathfrak{a} \subseteq \mathbf{k}Q$ is *admissible* if $\mathfrak{R}_Q^m \subseteq \mathfrak{a} \subseteq \mathfrak{R}_Q^2$ for some $m \geq 0$. In that case we call $A = \mathbf{k}Q/\mathfrak{a}$ a quiver algebra (of quiver Q with relations $\mathfrak{a}$)

Example

$$Q : \begin{array}{ccc} \overset{1}{\bullet} & \xrightarrow{\alpha} & \overset{2}{\bullet} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ \overset{3}{\bullet} & \xrightarrow{\delta} & \overset{4}{\bullet} \end{array}$$

$$\mathfrak{a} = \langle \beta\alpha - \delta\gamma \rangle$$

Then $(\mathbf{k}Q/\mathfrak{a})$-modules are precisely the representations of Q for which the square is commutative.

Definition

Rings R, S are *Morita equivalent* if R-Mod and S-Mod are equivalent as additive categories.
(If R, S are fin.-dim. algebras, the same is true for the full subcategories of fin.-dim. modules.)

Theorem (Gabriel)

Let A be any finite-dimensional algebra over an algebraically closed field. Then A is Morita equivalent to a bounded quiver algebra (of a finite quiver).

In particular, to provide a classification of bounded quiver algebras to finite/tame/wild type amounts to classsifying all finite-dimensional algebras.