Online homework 4, Problem 9

For each of the following equations, find the most general function $N(x, y)$ so that the equation is exact.

(a) $(3y\sin(3xy) + e^x)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$

(b) $(ye^{8xy} - 8x^4y + 2)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$

Solution:

(a) We have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(3y\sin(3xy) + e^x) = 3\sin(3xy) + 9xy\cos(3xy),$$

and the condition guaranteeing exactness is that this is equal to $\frac{\partial N}{\partial x}$. Thus, we obtain the condition

$$\frac{\partial N}{\partial x} = 3\sin(3xy) + 9xy\cos(3xy),$$

and we may thus recover $N(x, y)$ by integrating this expression with respect to x. That is,

$$N(x, y) = \int (3\sin(3xy) + 9xy\cos(3xy))\mathrm{d}x.$$

What remains is to evaluate the integral. We have $\int 3\sin(3xy)\mathrm{d}x = \frac{-\cos 3xy}{y}$, and to compute $\int 9xy\cos(3xy)\mathrm{d}x$, we integrate by parts ($u = 9xy$ and $\mathrm{d}v = \cos(3xy)$, so that $\mathrm{d}u = 9y$ and $v = \frac{\sin(3xy)}{3y}$):

$$\int 9xy\cos(3xy)\mathrm{d}x = 9xy \cdot \frac{\sin(3xy)}{3y} - \int 3\sin(3xy)\mathrm{d}x = 9xy \cdot \frac{\sin(3xy)}{3y} + \frac{\cos(3xy)}{y}.$$

Thus, we have

$$N(x, y) = \frac{-\cos 3xy}{y} + 9xy \cdot \frac{\sin(3xy)}{3y} + \frac{\cos(3xy)}{y} + g(y) = 9xy \cdot \frac{\sin(3xy)}{3y} + g(y),$$

where $g(y)$ is an arbitrary function depending only on y.

(b) We proceed exactly as in the previous part. We have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(ye^{8xy} - 8x^4y + 2) = e^{8xy} + 8xye^{8xy} - 8x^4,$$

hence

$$N(x, y) = \int (e^{8xy} + 8xye^{8xy} - 8x^4)\mathrm{d}x = -\frac{8}{5}x^5 + \frac{1}{8y}\int (e^u + ue^u)\mathrm{d}u,$$

where the substitution $u = 8xy$ was used (so that $\mathrm{d}u = 8y\mathrm{d}x$). The integral (antiderivative) of $e^u + ue^u$ is ue^u (check the derivative of this, or to deduce it, integrate by parts again), so we have

$$N(x, y) = -\frac{8}{5}x^5 + \frac{1}{8y}(8xy)e^{8xy} + g(y) = xe^{8xy} - \frac{8}{5}x^5 + g(y),$$

where $g(y)$ is again an arbitrary function depending only on y.