# Some problems on first order linear systems with complex eigenvalues.

**Problem 35-2**. Find a general solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} -10 & -10 & 10 \\ 5 & 6 & -4 \\ -12 & -10 & 12 \end{bmatrix} \mathbf{x}(t) \ .$$

*Solution:* First we find the eigenvalues and eigenvectors. We have

$$\begin{vmatrix} -10-\lambda & -10 & 10 \\ 5 & 6-\lambda & -4 \\ -12 & -10 & 12-\lambda \end{vmatrix} =$$
$$= (-10-\lambda)(-12-\lambda)(6-\lambda) - 500 - 480 + 120(6-\lambda) - 40(-10-\lambda) + 50(12-\lambda) =$$
$$= +\lambda^3 + 8\lambda^2 - 22\lambda + 20 \ .$$

Upon trying low values as roots, one checks that $\pm 1$ does not work, but $\lambda_1 = 2$ does. Dividing the characteristic polynomial by $(\lambda - 2)$ yields

$$-\lambda^2 + 6\lambda - 10 = -(\lambda^2 - 6\lambda + 10),$$

so the remaining two eigenvalues are

$$\lambda_{2,3} = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i.$$

Let us find the eigenvectors. For $\lambda = 2$, we have

$$\begin{bmatrix} -12 & -10 & 10 \\ 5 & 4 & -4 \\ -12 & -10 & 10 \end{bmatrix} \sim \begin{bmatrix} -12 & -10 & 10 \\ 5 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & -5 & 5 \\ 5 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 5 & 4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and thus, an eigenvector is of the form $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda = 3 + i$, we have

$$\begin{bmatrix} -13-i & -10 & 10 \\ 5 & 3-i & -4 \\ -12 & -10 & 9-i \end{bmatrix} \sim \begin{bmatrix} -13-i & -10 & 10 \\ 5 & 3-i & -4 \\ -2 & -4-2i & 1-i \end{bmatrix} \sim \begin{bmatrix} -13-i & -10 & 10 \\ 1 & -5-5i & -2-2i \\ -2 & -4-2i & 1-i \end{bmatrix} \sim$$
$$\begin{bmatrix} -13-i & -10 & 10 \\ 1 & -5-5i & -2-2i \\ 0 & -14-12i & -3-5i \end{bmatrix} \sim \begin{bmatrix} 0 & -70-70i & -14-28i \\ 1 & -5-5i & -2-2i \\ 0 & -14-12i & -3-5i \end{bmatrix} \sim \begin{bmatrix} 0 & 10+10i & 2+4i \\ 1 & -5-5i & -2-2i \\ 0 & 14+12i & 3+5i \end{bmatrix} \sim$$
$$\begin{bmatrix} 0 & 1 & \frac{3}{10}+\frac{1}{10}i \\ 1 & -5-5i & -2-2i \\ 0 & 1 & \frac{3}{10}+\frac{1}{10}i \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & \frac{3}{10}+\frac{1}{10}i \\ 1 & -5-5i & -2-2i \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5-5i & -2-2i \\ 0 & 1 & \frac{3}{10}+\frac{1}{10}i \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{3}{10}+\frac{1}{10}i \\ 0 & 0 & 0 \end{bmatrix} .$$

(In the first step on the third line, we divided first and third row by $10+10i, 14+12i$, resp.)

Thus, the eigenvector is of the form

$$\mathbf{x} = \begin{bmatrix} 1 \\ -\frac{3}{10} - \frac{1}{10}i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{10} \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -\frac{1}{10} \\ 0 \end{bmatrix},$$

or, after rescaling by 10,

$$\mathbf{x} = \begin{bmatrix} 10 \\ -3-i \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

For $\lambda = 3 - i$, we don't need to repeat the process. We just note that the eigenvalue is complex conjugate to the previous one, hence (since our matrix has only real entries) the corresponding eigenvector will be conjugate of the one we just found, i.e.

$$\mathbf{x} = \begin{bmatrix} 10 \\ -3+i \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 10 \end{bmatrix} - i \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Thus, the general solution should be of the form

$$y = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \left( e^{3t}\cos(t) \begin{bmatrix} 10 \\ -3 \\ 10 \end{bmatrix} - e^{3t}\sin(t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) + c_3 \left( e^{3t}\sin(t) \begin{bmatrix} 10 \\ -3 \\ 10 \end{bmatrix} + e^{3t}\cos(t) \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) =$$

$$= c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 10\cos(t) \\ -3\cos(t) + \sin(t) \\ 10\cos(t) \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 10\sin(t) \\ -3\sin(t) - \cos(t) \\ 10\sin(t) \end{bmatrix}.$$

**Warning:** The above expression is far from being unique. The reason is that the chosen eigenvectors are only unique up to multiplication by a complex number, which makes the vecors look quite differently. For example, another eigenvector for $\lambda = 3 + i$ is the vector

$$\tilde{\mathbf{x}} = \frac{1}{-3-i}\mathbf{x} = \begin{bmatrix} -3+i \\ 1 \\ -3+i \end{bmatrix}.$$

You can check for yourself that using this vector leads to a quite different expression (however, the space of all vector functions that can be described by those two expressions will be the same).

**Problem 35-5 (a)**. Find the solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} -7 & -1 \\ 17 & -5 \end{bmatrix} \mathbf{x}(t)$$

that satisfies the initial condition

$$\mathbf{x}(0) = \begin{bmatrix} -16 \\ 0 \end{bmatrix}.$$

*Solution:* Again, we find eigenvalues and eigenvectors first. We have

$$\begin{vmatrix} -7-\lambda & -1 \\ 17 & -5-\lambda \end{vmatrix} = (-7-\lambda)(-5-\lambda)+17 = \lambda^2+12\lambda+52,$$

hence

$$\lambda_{1,2} = \frac{-12 \pm \sqrt{144-4\cdot 52}}{2} = -6 \pm 4i.$$

To find eigenvector for $-6+4i$, we have

$$\begin{bmatrix} -1-4i & -1 \\ 17 & 1-4i \end{bmatrix} \sim \begin{bmatrix} -1-4i & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1+4i & 1 \\ 0 & 0 \end{bmatrix},$$

and hence a convenient form of the eigenvector is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1-4i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -4 \end{bmatrix}.$$

(As before, for the conjugate eigenvalue we get the conjugate eigenvector.) Thus, the general solution is of the form

$$y = c_1 e^{6t} \left( \cos(4t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin(4t) \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right) + c_2 e^{6t} \left( \sin(4t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos(4t) \begin{bmatrix} 0 \\ -4 \end{bmatrix} \right) =$$

$$= c_1 e^{6t} \begin{bmatrix} \cos(4t) \\ -\cos(4t)+4\sin(4t) \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} \sin(4t) \\ -\sin(4t)-4\cos(4t) \end{bmatrix}.$$

To find $c_1, c_2$ such that the resulting solution satisfies the given initial condition, as usual we plug in. At $t=0$,we have to have

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -16 \\ 0 \end{bmatrix}.$$

Looking at the first component yields $c_1 = -16$, while second component gives the equation $-c_1 - 4c_2 = 0$, so that $c_2 = 4$. Thus, the solution to the initial value problem is

$$y = e^{6t} \left( -16 \begin{bmatrix} \cos(4t) \\ -\cos(4t)+4\sin(4t) \end{bmatrix} + 4 \begin{bmatrix} \sin(4t) \\ -\sin(4t)-4\cos(4t) \end{bmatrix} \right) = e^{6t} \begin{bmatrix} -16\cos(4t)+4\sin(4t) \\ -68\sin(4t) \end{bmatrix}.$$