Quiz 2 Solution

1. Find the general solution to the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 2x^2 \quad .$$

Solution: The equation is a linear differential equation. We rewrite it to its standard form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x} \cdot y = 2x^2 \quad ,$$

and solve by the method of integrating factors. The integrating factor is

$$\mu(x) = e^{\int (-\frac{1}{x})\mathrm{d}x} = e^{-\ln x} = \frac{1}{x},$$

and so the solution is given by

$$y = \frac{\int \mu(x) \cdot 2x^2 \mathrm{d}x}{\mu(x)} = \frac{\int 2x \mathrm{d}x}{\frac{1}{x}} = x\left(x^2 + C\right) = x^3 + Cx,$$

where C is an arbitrary constant.

[Remark: in this solution, I am assuming $x > 0$ so that $\int \frac{1}{x}\mathrm{d}x = \ln x$. In case you used the more general $\int \frac{1}{x}\mathrm{d}x = \ln |x|$, this is also good (in fact, better), and ultimately leads to the same result; however, the manipulation with the absolute values might be more complicated.]

2. Determine whether the equation

$$(10xy^3 - 3ye^{xy})\mathrm{d}x + (15x^2y^2 - 3xe^{xy})\mathrm{d}y = 0$$

is exact, and if it is, solve it.

Solution: We apply the test for exactness, i.e. compute $\frac{\partial}{\partial y}(10xy^3 - 3ye^{xy})$, $\frac{\partial}{\partial x}(15x^2y^2 - 3xe^{xy})$ and compare them. We have

$$\frac{\partial}{\partial y}(10xy^3 - 3ye^{xy}) = 30xy^2 - 3e^{xy} - 3xye^{xy},$$

$$\frac{\partial}{\partial x}(15x^2y^2 - 3xe^{xy}) = 30xy^2 - 3e^{xy} - 3xye^{xy}.$$

These two are equal, so the equation is exact.

To solve the equation, we have to find the "potential function" $F(x, y)$, that is, a function with $\frac{\partial}{\partial x}F(x, y) = 10xy^3 - 3ye^{xy}$ and $\frac{\partial}{\partial y}F(x, y) = 15x^2y^2 - 3xe^{xy}$.

Integrating the first expression with respect to x yields

$$F(x, y) = \int 10xy^3 - 3ye^{xy}\mathrm{d}x = 5x^2y^3 - 3e^{xy} + g(y)$$

where $g(y)$ is a function depending possibly on y. To determine $g(y)$, we use the second condition on partial derivatives of F. We have

$$15x^2y^2 - 3xe^{xy} = \frac{\partial}{\partial y}F(x, y) = \frac{\partial}{\partial y}\left(5x^2y^3 - 3e^{xy} + g(y)\right) = 15x^2y^2 - 3xe^{xy} + \frac{\mathrm{d}\,g}{\mathrm{d}\,y},$$

so $\frac{\mathrm{d}\,g}{\mathrm{d}\,y} = 0$ and $g(y)$ is thus an arbitrary constant, we may choose e.g. $g(y) = 0$. Thus, $F(x, y) = 5x^2y^3 - 3e^{xy}$, and the (implicit) solutions to the equation are of the form

$$5x^2y^3 - 3e^{xy} = C, \quad C \text{ an arbitrary constant.}$$