Quiz 3 Solution

1. Find the general solution to the equation

$$5y' - y = y^{-4}.$$

Solution: This is a Bernoulli equation with y^{-4} on the right-hand side, and so we use the substitution $v = y^{1-(-4)} = y^5$. That is, we have $y = y^{-4}v$, and

$$v' = (y^5)' = 5y^4y', \text{ i.e. } y' = \frac{1}{5}y^{-4}v'.$$

Plugging these expressions for y and y' in the original equation yields

$$y^{-4}v' - y^{-4}v = y^{-4},$$

so we may cancel y^{-4} and obtain the simplified equation

$$v' - v = 1.$$

This is a linear equation which we may solve e.g. by the method of integrating factors: we have

$$\mu(x) = e^{\int(-1)\mathrm{d}x} = e^{-x},$$

hence

$$v = \frac{\int 1 \cdot e^{-x}\mathrm{d}x}{e^{-x}} = e^x(-e^{-x} + C) = Ce^x - 1.$$

Since $y = \sqrt[5]{v}$, we have

$$y = \sqrt[5]{Ce^x - 1},$$

where C is an arbitrary constant.

2. An object of mass 1 kg starts falling through the air. The air resistance force (in Newtons) is $-2v$ where v is the velocity of the object, the acceleration due to gravity is assumed to be $10\,\mathrm{m/sec}^2$. Determine the equation of the motion of the object (i.e. the equation describing the position $x(t)$).
[The initial velocity and position are both assumed to be 0.]

Solution: The equation from Newton's law of motion has in this case the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - 2v$$

(where mg is the force due to gravity and $-2v$ is the force of air resistance given in the problem statement). Plugging in $m = 1\,\mathrm{kg}$ and $g = 10\,\mathrm{m/sec}^2$ yields the linear differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 2v\ ,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + 2v = 10\ .$$

We can solve this either by separation of variables, or by the integrating factor method. Let us use the integrating factor. The factor is

$$\mu(t) = e^{\int 2\mathrm{d}t} = e^{2t},$$

so the solution v of the equation is

$$v(t) = \frac{\int 10e^{2t}\mathrm{d}t}{e^{2t}} = \left(5e^{2t} + C\right)e^{-2t} = 5 + Ce^{-2t} \; .$$

Plugging in the initial condition $v(0) = 0$ (meaning that at the beginning of the fall, the velocity of the dropped object is 0) yields $C = -5$. Thus,

$$v(t) = 5 - 5e^{-2t}.$$

To get the equation of the motion, i.e. to determine te position function $x(t)$, we integrate the velocity with respect to time, i.e.

$$x(t) = \int_0^t v(s)\mathrm{d}s = \int_0^t (5 - 5e^{-2s})\mathrm{d}s = \left[5s + \frac{5}{2}e^{-2s}\right]_0^t = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2} \; .$$