Quiz 4 Solution

1. Find a solution to the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ 2x_2 + x_3 &= -1 \\ -x_1 \qquad + x_3 &= 0 \ . \end{aligned}$$

Solution:
An elementary solution: Third equation can be rewritten as $x_1 = x_3$, so we may plug in x_1 for x_3 in the first two equations and obtain

$$\begin{aligned} 3x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= -1 \ . \end{aligned}$$

Subtracting the second equation from the first one three times gives $-5x_2 = 5$, so $x_2 = -1$. Plugging this into the first equation gives $3x_1 - 1 = 2$, so $x_1 = 1$, and so the solution is $x_1 = 1, x_2 = -1, x_3 = x_1 = 1$.
A systematic solution: We use row operations on the (augmented) matrix of the system. We have

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & -1 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim_1 \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim_2 \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim_3 \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -5 & -5 \end{array}\right] \sim_4 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim_5$$

$$\sim_5 \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim_6 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(where
$\sim_1$=swapping the rows appropriately,
$\sim_2$=adding the first row to the second 2 times,
$\sim_3$=subtracting the second row from the third 2 times,
$\sim_4$=dividing the third row by -5 and the first row by -1,
$\sim_5$=subtracting the third row from the second 3 times,
$\sim_6$=addding the third row to the first). Thus, we obtain the same solution, i.e. $x_1 = 1, x_2 = -1, x_3 = 1$.

2. Determine whether the vector $\begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$ is a linear combination of the vectors $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, and justify.

Solution: To decide whether the first vector is a linear combination of the other two is the same as to decide whether the vector equation

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix} \qquad (1)$$

has a solution (x_1, x_2). By row operations, we may reduce the associated augmented matrix of the system as follows:

$$\left[\begin{array}{cc|c} -1 & -2 & 0 \\ 2 & 1 & 6 \\ 3 & 2 & 8 \end{array}\right] \sim_0 \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 6 \\ 3 & 2 & 8 \end{array}\right] \sim_1 \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 3 & 2 & 8 \end{array}\right] \sim_2 \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 0 & -4 & 8 \end{array}\right] \sim_3$$

$$\sim_3 \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right] \sim_4 \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \sim_5 \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

(where
$\sim_0$=multiplying the first row by (-1),
$\sim_1$=subtracting the first row from the second 2 times,
$\sim_2$=subtracting the first row from the third 3 times,
$\sim_3$=dividing the second row by -3 and the third row by -4,
$\sim_4$=subtracting the second row from the third,
$\sim_5$=subtracting the second row from the first 2 times).

We see that the system has a solution $x_1 = 4, x_2 = -2$, and so the vector $\begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix}$ **is** a linear combination of the vectors $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix},$.

[*Remark:* If we are not interested in the coefficients x_1, x_2, we could have stop the computation e.g. after $\sim_4$: The key information is that the right-hand side column is not the pivot column of the augmented matrix.]