Quiz 7 Solution

1. Determine whether $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the span of the vectors $\begin{bmatrix}3\\1\\4\end{bmatrix}$ and $\begin{bmatrix}-1\\3\\2\end{bmatrix}$.

Solution:

To be in the span is equivalent to the statement that the vector equation

$$c_1 \begin{bmatrix}3\\1\\4\end{bmatrix} + c_2 \begin{bmatrix}-1\\3\\2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

has a solution, so this is what we have to check. We have

$$\left[\begin{array}{cc|c}3 & -1 & 1\\1 & 3 & 2\\4 & 2 & 3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 2\\3 & -1 & 1\\4 & 2 & 3\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 2\\0 & -10 & -5\\0 & -10 & -5\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 2\\0 & -10 & -5\\0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 3 & 2\\0 & 1 & \frac{1}{2}\\0 & 0 & 0\end{array}\right].$$

Thus, the system is consistent (the right-hand-side column is not a pivot column), hence the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the span of $\begin{bmatrix}3\\1\\4\end{bmatrix}$ and $\begin{bmatrix}-1\\3\\2\end{bmatrix}$.

2. Find a basis of the null space of the matrix

$$\begin{bmatrix}-1 & 4 & 2 & 5\\0 & 1 & 1 & 1\\3 & -2 & 4 & -5\end{bmatrix}.$$

Solution: We are just solving the system of homogeneous euations determined by the given matrix. After few row operations, we obtain the row echelon form

$$\begin{bmatrix}-1 & 4 & 2 & 5\\0 & 1 & 1 & 1\\3 & -2 & 4 & -5\end{bmatrix} \sim \begin{bmatrix}1 & -4 & -2 & -5\\0 & 1 & 1 & 1\\3 & -2 & 4 & -5\end{bmatrix} \sim \begin{bmatrix}1 & -4 & -2 & -5\\0 & 1 & 1 & 1\\0 & 10 & 10 & 10\end{bmatrix} \sim \begin{bmatrix}1 & -4 & -2 & -5\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -1\\0 & 1 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix},$$

and so the free variables are x_3 and x_4 (that is, third and fourth column are not pivot columns). Treating them as parameters, i.e. setting $x_3 = r$, $x_4 = s$, we have $x_2 = -r - s$, $x_1 = -2r + s$. That is, all the solutions (i.e. all the vectors in the null space) are of the form

$$\begin{bmatrix}-2r+s\\-r-s\\r\\s\end{bmatrix} = r\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}.$$

The basis of the nullspace is thus

$$\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}.$$