Quiz 8 Solution

Given a matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & -2 & 1 \\ 5 & 4 & 4 \end{bmatrix},$$

1. Find a basis of $\operatorname{Nul} A$, and a basis of $\operatorname{Col} A$.

   Solution: For both we use row operations to get to the row echelon form:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & -2 & 1 \\ 5 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -1 \\ 0 & -6 & -1 \\ 0 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -6 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that the only free variable is x_3, and when $x_3 = r$, the corresponding solution is

$$\begin{bmatrix} -\frac{2}{3}r \\ -\frac{1}{6}r \\ r \end{bmatrix} = r \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{6} \\ 1 \end{bmatrix},$$

so the basis consists of one vector only, which can be taken as

$$\mathbf{v} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{6} \\ 1 \end{bmatrix}$$

(or, to get rid of fractions, we may take $6\mathbf{v} = \begin{bmatrix} -4 \\ -1 \\ 6 \end{bmatrix}$). The basis of $\operatorname{Col} A$ can be picked out of the columns of A, and the recipe is to pick out the pivot columns. From the row echelon form we see that the first two columns are pivot columns, hence a basis of the column space is

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}.$$

2. Find $\operatorname{rank} A, \dim \operatorname{Nul} A, \dim \operatorname{Col} A$ and $\dim \operatorname{Row} A$.

   Solution: We have $\dim \operatorname{Nul} A = 1$ since it has a basis consisting of one vector. What we always have is that $\operatorname{rank} A = \dim \operatorname{Col} A = \dim \operatorname{Row} A$, and we just saw that $\dim \operatorname{Col} A = 2$ (since its basis consists of two vectors). So we have

$$\dim \operatorname{Nul} A = 1, \quad \operatorname{rank} A = \dim \operatorname{Col} A = \dim \operatorname{Row} A = 2.$$