02.11.22 Lesson 12

Chain Rule & Derivative of Natural log

Warm up

Last time Chain Rule w/ power rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

works the same with other functions $u = g(x)$

$$\frac{d}{dx}[\sin u] = \cos u \,\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = -\sin u \,\frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = e^u \,\frac{du}{dx}$$

← ☆ Now we have to derive the function in the exponent ☆

New

$$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x} \qquad \frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} \text{ or } \frac{u'}{u}}$$

Example 1 Find the Derivatives

(a) $y = \cos(x^2)$

Let $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\sin(x^2)\, 2x$$

$$= -2x\sin(x^2)$$

(b) $y = e^{3x+1}$

Let $u = 3x+1$

$$\frac{dy}{dx} = e^{3x+1}\frac{du}{dx}$$

$$= e^{3x+1}(3)$$

$$= 3e^{3x+1}$$

Example 2

$$y = \underbrace{(3x-2)^3}_{f}\underbrace{(x^2+4)^2}_{g}$$

Find $y'(1)$

$$y' = f'g + fg'$$

Derive the inside

$$f' = 3(3x-2)^2(3)$$

$$= 9(3x-2)^2$$

$$g' = 2(x^2+4)(2x)$$

$$= 4x(x^2+4)$$

$$y' = \overbrace{9(3x-2)^2}^{f'}\overbrace{(x^2+4)^2}^{g} + \underbrace{3(x-2)^3}_{f}\underbrace{(x^2+4)^2}_{g'}$$

$$y'(1) = 9(1)^2(5)^2 + 3(-1)^3(5)^2$$

$$= 9\cdot 25 - 3\cdot 25 = 6\cdot 25 = 150$$

Example 3 Let $h(x) = \frac{\sqrt{7-x^2}}{3x}$ $\frac{f}{g}$ Find $h'(-1)$

$$h' = \frac{\overbrace{\tfrac{1}{2}(7-x^2)^{-1/2}(-2x)}^{f' \text{ w/ chain rule}}\overbrace{(3x)}^{g} - \sqrt{7-x^2}\,(3)}{(3x)^2} \leftarrow g^2$$

$$= \frac{(7-(-1)^2)^{-1/2}(-(-1))(3(-1)) - \sqrt{7-(-1)^2}\,(3)}{(3(-1))^2}$$

$$= \frac{\frac{-3}{\sqrt{6}} - \sqrt{6}(3)}{9} = \frac{\frac{3}{\sqrt{6}} - \sqrt{6}(3)\frac{\sqrt{6}}{\sqrt{6}}}{9}$$

$$= \frac{-3-6(3)}{\sqrt{6}} \cdot \frac{1}{9} = \frac{-21}{9\sqrt{6}} = -\frac{7}{3\sqrt{6}}$$

Example 4 $y = 11\sec^2(2x)$ Find $y'(\frac{\pi}{2})$

$$y' = 11 \cdot 2(\underbrace{\sec(2x)}_{\text{keep inside}})(\underbrace{\sec(2x)\tan(2x)}_{\frac{d}{dx}[\sec(2x)]})(2)$$

derivative of inside

$$y' = 44\sec^2(2x)\tan(2x)$$

$\tan(\pi) = 0$

$$y'\left(\frac{\pi}{2}\right) = 44\left(\sec\left(2\frac{\pi}{2}\right)\right)^2 \tan\left(2\frac{\pi}{2}\right) = 0$$

Example 5 $\quad y = 5\sin^2(3x)$ $\quad$ Find $y'(\frac{\pi}{3})$

$$y' = 5(2\cdot \underbrace{\sin(3x)}_{\text{Keep inside}})\ \underbrace{\cos(3x)\ 3}_{\frac{d}{dx}[\sin(3x)]}$$

$$y' = 30\sin(3x)\cos(3x)$$

$$y'\left(\frac{\pi}{3}\right) = 30\ \underbrace{\sin(\pi)}_{=0}\ \underbrace{\cos(\pi)}_{=-1} = 0$$

Example 6 $\quad f(x) = e^{3x}\tan(2x)$ $\quad$ find f'

$$f'(x) = \overset{h'}{3e^{3x}}\ \overset{g}{\tan(2x)} + \overset{h}{e^{3x}}\ \overset{g}{\sec^2(2x)(2)}$$

$$= e^{3x}(3\tan(2x) + 2\sec^2(2x))$$

Example 7 $\quad f(x) = \sqrt{3x}\ \ln(4x)$ $\quad \frac{d}{dx}[\ln(4x)]$

$$f'(x) = \overset{h'}{\tfrac{1}{2}(3x)^{-1/2}(3)}\ \overset{g}{\ln(4x)} + \overset{h}{\sqrt{3x}}\ \overset{g'}{\frac{1}{4x}(4)}$$

$$= \frac{3}{2}\frac{1}{\sqrt{3x}}\ln(4x) + \frac{\sqrt{3x}}{x}$$

Example 8 $\quad Y = \ln\sqrt{\dfrac{3x+2}{x^2-2}}$ find $y'(0)$

$$Y = \frac{1}{2}\ln\left(\frac{3x+2}{x^2-2}\right) = \frac{1}{2}\left(\ln(3x+2) - \ln(x^2-2)\right)$$

$$Y' = \frac{1}{2}\left(\frac{3}{3x+2} - \frac{2x}{x^2-2}\right)$$

$$Y'(0) = \frac{1}{2}\left(\frac{3}{3(0)+2} - \frac{2(0)}{(0)^2-2}\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

Example 9 $\quad g(x) = \dfrac{3\ln x}{7-3x}$ $\quad g(e)$

$$g'(x) = \frac{(7-3x)\,3\,\frac{1}{x} - 3\ln x\,(-3)}{(7-3x)^2}$$

$$= \frac{(7-3e)\frac{3}{e} + 9\ln(e)}{(7-3e)^2} = \frac{\frac{21}{e} - \cancel{9} + \cancel{9}}{(7-3e)^2}$$

$$= \frac{21}{e(7-3e)^2}$$

Example 10

At sea level, air pressure is 30 inches of mercury. At an altitude of h feet above sea level, the air pressure, P, in inches of mercury is given by the function

$$P = 20 e^{-0.000021h}$$

Determine the rate of change of the air pressure

$$P' = 20(-0.000021) e^{-0.000021h}$$

$$= -0.00042 e^{-0.000021h}$$