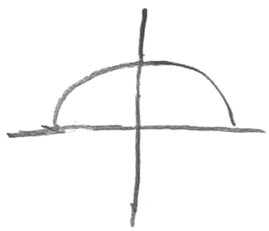
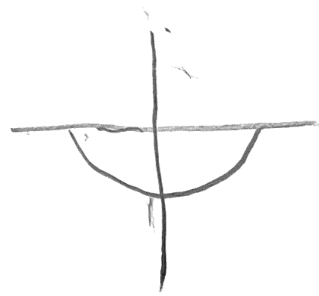
02.16.22 Lesson 14 Implicit Differentiation

How do you differentiate the equation we describe the circle with? $x^2+y^2=9$

Explicit form $\rightarrow$ $y=\sqrt{9-x^2}$ or $y=-\sqrt{9-x^2}$





Note: with some equations its way too difficult to write as an explicit function of y.

$\underline{y=f(x)}$

$\underline{\text{Explicit form}}$

Theres an easier / different way

implicit form

$$x^2+y^2=9$$

$$\frac{d}{dx}[x^2+y^2]=\frac{d}{dx}[9]$$

$$2x+\underbrace{2y\,y'}_{\text{Chain rule to } y^2}=0$$

$$2y\,y'=-2x$$

$$y'=\frac{-2x}{2y}=-\frac{x}{y}$$

We call this implicit differentiation

Example 1

$$2xy + 7y^2 = 6 + 3x^2$$

use implicit differentiation to find $\frac{dy}{dx}$

$$\frac{d}{dx}[2xy + 7y^2] = \frac{d}{dx}[6 + 3x^2]$$

$$\underbrace{2xy' + 2y}_{\text{product rule}} + \underbrace{14yy'}_{\text{chain rule with } y^2} = 6x$$

$$2xy' + 14yy' = 6x - 2y$$

$$y'(2x + 14y) = 6x - 2y$$

$$\frac{dy}{dx} = y' = \frac{6x - 2y}{2x + 14y}$$

Idea Every thing is the same with our derivative rules, but because y is a function of x, when ever we apply a derivative rule to y, we have to write y' or $\frac{dy}{dx}$ because a chain rule is actually happening.

If $y = 3x^2 + 7$ and we want to find the derivative of y^2 or yx we need the chain rule.

$$\frac{d}{dx}[y^2] = 2y\frac{dy}{dx} = 2y(6x)$$

$$\frac{d}{dx}[yx] = y(1) + y'x$$

$$= \underbrace{(3x^2+7)}_{y} + \underbrace{(6x)}_{y'}x$$

Example 2 Use implicit differentiation to find the slope of the tangent line to the graph of at $(2,1)$

$$\frac{3}{x} + \frac{1}{2y} = 19$$

remember to derive both sides

$$\frac{d}{dx}\left[\frac{3}{x} + \frac{1}{2y}\right] = \frac{d}{dx}[19]$$

$$\frac{d}{dx}\left[3x^{-1} + \frac{1}{2}y^{-1}\right] = \frac{d}{dx}[19]$$

$$-3x^{-2} - \frac{1}{2}y^{-2}y' = 0$$

$$-\frac{1}{2}y^{-2}y' = 3x^{-2}$$

$$y' = \frac{-6x^{-2}}{y^{-2}} = -6\frac{y^2}{x^2}$$

Example 2 ... cont.

plug in (2,1)

$$\frac{dy}{dx} = -6\,\frac{1^2}{2^2} = \frac{-6}{4} = -\frac{3}{2}$$

Example 3 use implicit differentiation to find

$\frac{dy}{dx}$ given $3\cos(x+y)\sin(x) = 7$

$$\frac{d}{dx}\left[3\cos(x+y)\sin(x)\right] = \frac{d}{dx}[7]$$

product rule

$$3\cos(x+y)\cos x - 3\sin(x+y)\,y'\sin(x) = 0$$

solve for y'

$$\sin(x+y)\,y'\sin(x) = \cos(x+y)\cos(x)$$

$$y' = \frac{\cos(x+y)\cos(x)}{\sin(x+y)\sin(x)}$$

Example 4 use implicit differentiation to find $\frac{dy}{dx}$ given $3\tan\left(\frac{2x}{y}\right) = x^2$

$$\frac{d}{dx}\left[3\tan\left(\frac{2x}{y}\right)\right] = \frac{d}{dx}\left[x^2\right]$$

$$3\sec^2\left(\frac{2x}{y}\right)\left(\frac{2}{y} - \frac{2xy'}{y^2}\right) = 2x$$

$$\frac{d}{dx}\left[2xy^{-1}\right] = 2y^{-1} + 2x(-1y^{-2}y')$$

(product rule)

(distribute)

$$\frac{6}{y}\sec^2\left(\frac{2x}{y}\right) - \frac{6x}{y^2}\sec^2\left(\frac{2x}{y}\right)y' = 2x$$

$$-\frac{6x}{y^2}\sec^2\left(\frac{2x}{y}\right)y' = 2x - \frac{6}{y}\sec^2\left(\frac{2x}{y}\right)$$

$$y' = \frac{2x - \frac{6}{y}\sec^2\left(\frac{2x}{y}\right)}{-\frac{6x}{y^2}\sec^2\left(\frac{2x}{y}\right)}$$

$$= \frac{2xy^2 - 6y\sec^2\left(\frac{2x}{y}\right)}{-6x\sec^2\left(\frac{2x}{y}\right)}$$

Example 5 Find $\frac{dy}{dx}$ given $e^{3xy} = 8x$

$$\frac{d}{dx}\left[e^{3xy}\right] = \frac{d}{dx}[8x]$$

chain rule from exponent with product rule inside

$$e^{3xy}(3y + 3xy') = 8$$

$$e^{3xy}3y + e^{3xy}3xy' = 8$$

$$3xy'e^{3xy} = 8 - 3ye^{3xy}$$

$$y' = \frac{8 - 3ye^{3xy}}{3xe^{3xy}}$$