02.18.22

Lessons 14 Related Rates pt. 1

Last time implicit differentiation

Today & Monday Related Rates

Idea suppose a circle is growing and

we know the radius is changing at a rate of 5 cm/min. What is the rate of change of the area of the circle when $r = 4$ cm.

Recall rate of change = derivative!

Area of circle: $A = \pi r^2$

- rate of change of radius 5 cm/min means

$$\frac{dr}{dt} = 5 \text{ cm/min} \qquad \frac{\text{change in length}}{\text{change in time}}$$

we want $\dfrac{dA}{dt}$ $\qquad \dfrac{\text{Change in Area}}{\text{Change in time}}$

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continued!

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Why? r is a function of time $r(t)$

$$A(t) = \pi (r(t))^2$$

$$\frac{dA}{dt} = 2\pi r(t) \frac{dr}{dt}$$

Now $r = 4$ cm and $\frac{dr}{dt} = 5$ cm/min so we get

$$\frac{dA}{dt} = 2\pi (4\,cm)(5\,cm/min) = 40\pi \; cm^2/min$$

The area of the circle and its radias are related. As the radius of circle changes over time so does the area.

Example 2 Assume x and y are both differentiable functions of t and $x^3 y = 7$

Find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 3$ and $x = -1$.

Solution Think of $x^3 y = 7$ as

$$(x(t))^3 \, y(t) = 7$$

then use product rule and implicit differentiation

$$\frac{d}{dt}\left[(x(t))^3 \, y(t)\right] = \frac{d}{dt}[7]$$

$$\overbrace{3(x(t))^2 \frac{dx}{dt}}^{\text{chain rule}} y(t) + (x(t))^3 \frac{dy}{dt} = 0$$

change back to normal x and y

$$3x^2 \frac{dx}{dt} y + x^3 y \frac{dy}{dt} = 0$$

plug in $\frac{dx}{dt} = 3$ and $x = -1$ and y from

$$(-1)^3 y = 7$$
$$y = -7$$

$$3(-1)^2(3)(-7) + (-1)^3(-7)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{63}{7} = -9$$

Example 3 A spherical balloon is being inflated. The radius is increasing at a rate of 2 cm/sec.

How fast is the volume increasing when $r = 8$ cm?

The volume of a sphere is $V = \frac{4}{3}\pi r^3$

Solution

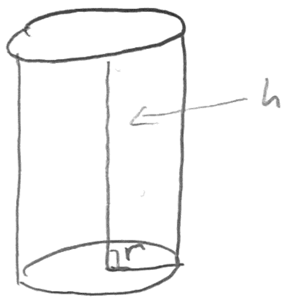
$$\frac{dV}{dt} = \frac{4}{3}\pi \underbrace{3r^2 \frac{dr}{dt}}_{\frac{d}{dt}\left[(r(t))^3\right]}$$

$$\frac{dV}{dt} = 4\pi (8)^2\, 2$$

$$= 512\pi \ \text{cm}^3/\text{sec}$$

Example 4

A cylindrical tank has a radius of 10 cm for the base. How fast does the water level in the tank drop when the water is being drained at 4 cm^3/sec?



Volume of cylinder:

$$V = \pi r^2 h$$

Given $\quad r = 10 \qquad \frac{dV}{dt} = 4 \, cm^3/sec$

why? "water being drained and units tell us rate of change of volue"

Need to find $\quad \frac{dh}{dt}$ $\quad$ Why?

"How fast does water level drop means" change in height.

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$$V = \pi r^2 h$$

$$V = 10^2 \pi h$$

$$\frac{d}{dt}[V] = \frac{d}{dt}[10^2 \pi h]$$

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

②

plug in

$$4 = 100\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{100\pi} \text{ cm/sec}$$

Note: In cone HW problem we are also given $\frac{dV}{dt}$ and have to solve for $\frac{dh}{dt}$

Extra practice

Surface area of balloon when rate of radius is 2 cm/sec and $r = 6$ cm

$SA = 4\pi r^2$