02.25.22 Lesson 18

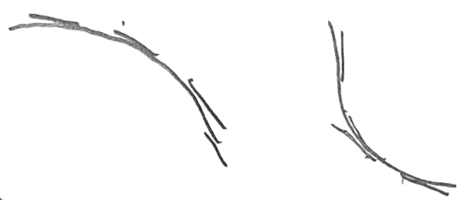
Increasing and Decreasing Functions & First Derivative Test

Increasing function

As x gets bigger, f(x) gets bigger

- $f'(x) > 0$, the slope of the tangent lines are positive when f(x) is increasing

Decreasing function



As x gets bigger,
f(x) gets smaller.

- $f'(x) < 0$, the slope of the tangent lines are negative when f(x) is decreasing.

Theorem: Let $f(x)$ be a continuous and differentiable function on an open interval I

- If $f'(x) > 0$ for all x in I, then $f(x)$ is <u>increasing</u> on I.

- If $f'(x) < 0$ for all x in I, then $f(x)$ is <u>decreasing</u> on I

Example 1 $f(x) = 3x^2 - 2x$

Find where $f(x)$ is increasing

- Set $f'(x) = 0$, find critical numbers

- do a number line test.

set

$$f'(x) = 6x - 2 = 0$$

$$x = \frac{2}{6} = \frac{1}{3}$$

$f'(x) < 0$ $\quad \frac{1}{3} \quad$ $f'(x) > 0$

test value: $x = 0$ $\quad$ $x = 1$

$f'(0) = -2$ $\quad$ $f'(1) = 4$

- f is <u>increasing</u> on $(\frac{1}{3}, \infty)$

- f is <u>decreasing</u> on $(-\infty, \frac{1}{3})$

Some things to Note:

- The actual test values do not matter as long as you pick one less than the critical number and one larger than the critical number.

- The sign of $f'(x)$ is whats important not what $f'(x)$ equals, i.e. is $f'(x) > 0$ or $f'(x) < 0$?

- * The sign of $f'(x)$ can only change at a C.N. *

The First Derivative Test

(a) $f'(x) > 0$ | $f'(x) < 0$; c — e.g. $\cap$ $\Rightarrow$ rel. max at $x = c$

(b) $f'(x) < 0$ | $f'(x) > 0$; c — e.g. $\cup$ $\Rightarrow$ rel. min at $x = c$

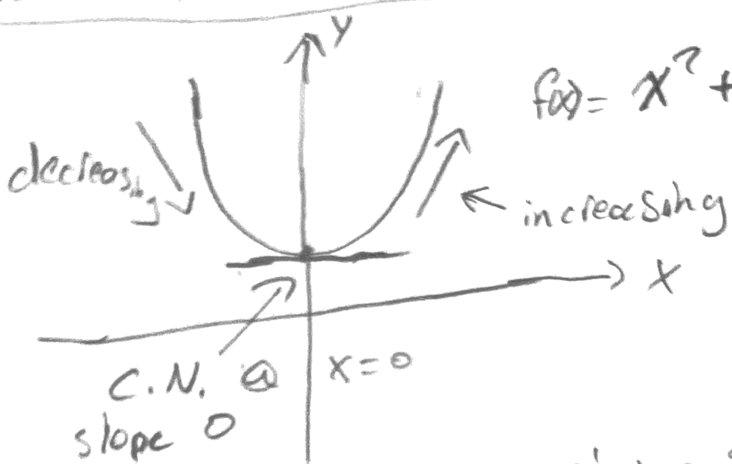
(c) $f'(x) > 0$ | $f'(x) > 0$; c — eg.  $\Rightarrow$ neither max or min

(d) $f'(x) < 0$ | $f'(x) < 0$; c — eg.  $\Rightarrow$ neither max or min

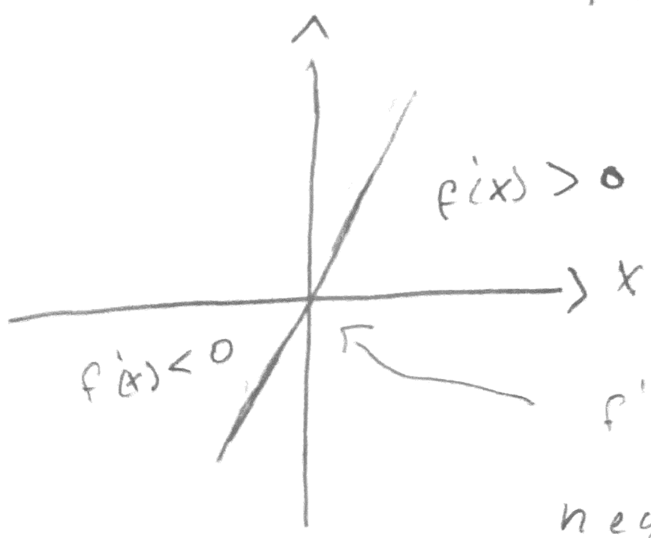
Note A critical Number is a rel. max or rel. min only if the derivative changes signs at that point $x = c$

- A picture for why this all works

$f(x) = x^2 + 2$



$f'(x) = 2x$



f'(x) changed from negative to positive so $x = 0$ is a <u>relative minimum</u>

Example 2 $\qquad f(x) = \frac{1}{3}x^3 + 3x^2 - 7x + 9$

Find where $f(x)$ is increasing and decreasing

Find where $f(x)$ has a rel. max and rel. min

Solution

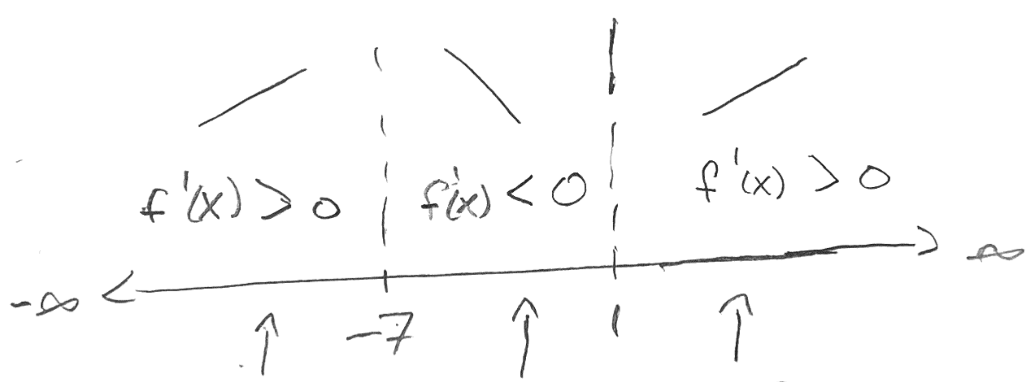
(1) Find the C.N.

(2) Use the first Derivative test to identify rel. max / rel. min

$$f'(x) = x^2 + 6x - 7 \overset{\text{set}}{=} 0$$

$$(x+7)(x-1) = 0$$

C.N. at $x = -7$ and $x = 1$



Test values: $x = -8$ $\qquad x = 0$ $\qquad x = 2$

- $f'(-8) = (-8+7)(-8-1)$
  $(-)(-) = (+)$

- $f'(0) = (7)(-1)$
  $(+)(-) = (-)$

- $f'(2) = (2+7)(2-1)$
  $(+)(+) = (+)$

Example 2 ... cont.

$f(x)$ is:

increasing: $(-\infty, 0)$, $(2, \infty)$

decreasing: $(-7, 1)$

rel. max; $x = -7$; so $(-7, f(-7)) = (-7, -\frac{22}{3})$

rel. min; $x = 1$; so $(1, f(1)) = (1, \frac{58}{3})$

Example 3 Let $g(x) = 3x^4 - 5x^3$

Find where $g(x)$ is increasing / decreasing

Find where $g(x)$ has rel. max and rel. min

Solution

- Find C.N.
- First Derivative Test

$$g(x) = 3x^4 - 5x^3$$

$$g'(x) = 12x^3 - 15x^2 = 0 \quad \text{set}$$

$$3x^2(4x - 5) = 0$$

C.N.: $x = 0$, $x = \frac{5}{4}$

Example 3 ... Cont.

$g'(x) < 0$ | $g'(x) < 0$ | $g'(x) > 0$

0 $\quad$ $\frac{5}{4}$

test values: $x = -1$ $\quad$ $x = 1$ $\quad$ $x = 2$

$$g'(-1) = 3(-1)^2(4(-1) - 5)$$
$$(+)\ (-) = (-)$$

$$g'(2) = 3(2)^2(4(2) - 5)$$
$$(+)\ (+) = (+)$$

$$g'(1) = 3(1)^2(4(1) - 5)$$
$$(+)\ (-) = (-)$$

$g(x)$ is

increasing: on $\left(\frac{5}{4}, \infty\right)$

Decreasing: on $\left(-\infty, \frac{5}{4}\right)$

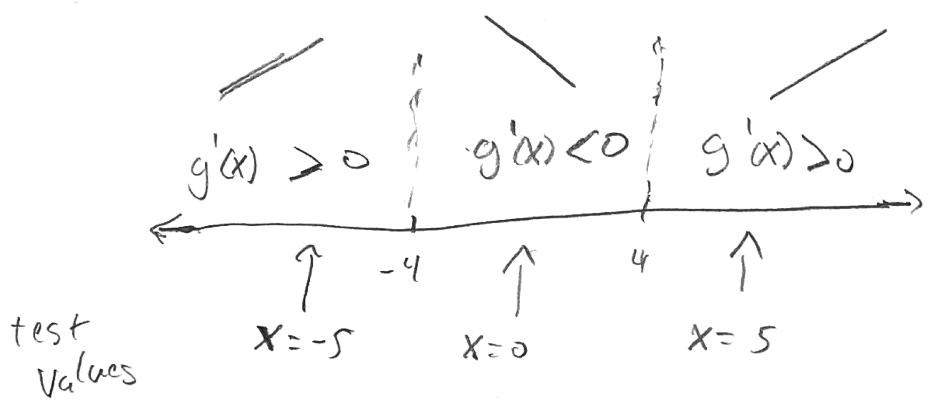
- $g(x)$ has a rel. min at $x = \frac{5}{4}$; $\left(\frac{5}{4}, g\left(\frac{5}{4}\right)\right) = \left(\frac{5}{4}, -2.441\right)$
- $x = 0$ is not a min or max

B/c the sign of $g'(x)$ does not change.

Example 4 If $g'(x) = e^{2x}(2x^2 - 32)$

Find where $g(x)$ is increasing, decreasing and where $g(x)$ has it's rel. max and rel. min.

C.N. at $x = \pm\sqrt{\frac{32}{2}} = \pm\sqrt{16} = \pm 4$



increasing: $(-\infty, -4)$ and $(4, \infty)$

decreasing: $(-4, 4)$

rel max: $x = -4$

rel min: $x = 4$

$g'(-5) = e^{-10}(2(25) - 32) = (+)$, $g'(0) = e^{0}(-32) = (-)$

$g'(5) = e^{10}(2(25) - 32) = (+)$

Example 5   Find the critical numbers of

$$f(x) = 2\sin(3x) + 3x$$

on the interval $(0, \pi)$. Then identify which x values are relative maximums.

Solution (1) Find the critical points

(2) Carry out the First Derivative Test

$$f'(x) = 2\cos(3x)\,3 + 3$$
$$= 6\cos(3x) + 3 = 0$$
$$\cos(3x) = -\frac{3}{6} = -\frac{1}{2}$$

Now since $\cos(\theta) = -\frac{1}{2}$ when $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$ we get

$$\theta = 3x = \frac{2\pi}{3} + 2n\pi \qquad n = \pm 1, \pm 2, \pm 3, \ldots$$

$$3x = \frac{4\pi}{3} + 2n\pi$$

Recall $\cos(\theta)$ is periodic and repeates itself in multiples of 2π.

But since we have $\cos(3x)$, it repeates itself every $\frac{2n\pi}{3}$ times insted.

Example 5 ... cont.

So our equations for x become

$$(1)\quad 3x = \frac{2\pi}{3} + 2n\pi \longrightarrow \cancel{x} = \frac{2\pi}{9} + \frac{2n\pi}{3} \quad (1)$$

$$(2)\quad 3x = \frac{4\pi}{3} + 2n\pi \longrightarrow x = \frac{4\pi}{9} + \frac{2n\pi}{3} \quad (2)$$

Plugging in values of n we will find thre C.N. in the interval $(0, \pi)$.

(1) $n = 0$ $\quad x = \frac{2\pi}{9}$

$n = 1$ $\quad x = \frac{2\pi}{9} + \frac{2\pi}{3} = \frac{8\pi}{9}$

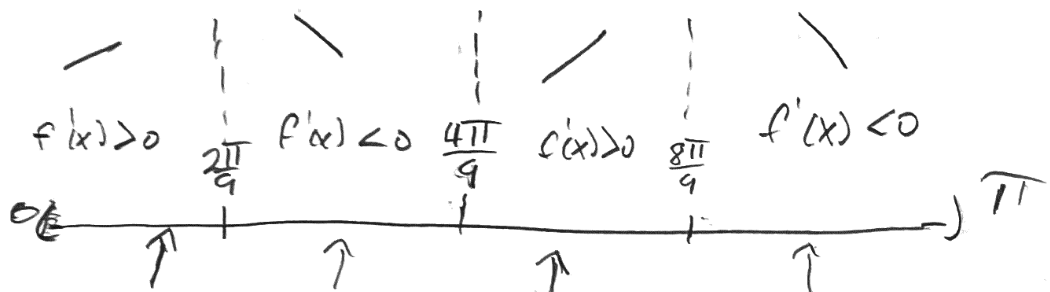
$\cancel{n} = 2$ $\quad x = \frac{2\pi}{9} + \frac{2(2)\pi}{3} = \frac{14\pi}{9} > \pi$ so outside our interval

(2) $n = 0$ $\quad x = \frac{4\pi}{9}$

$\cancel{n} = 1$ $\quad x = \frac{4\pi}{9} + \frac{2\pi}{3} = \frac{10\pi}{9} > \pi$ so outside our interval

So our C.N. are $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

Example 5 ... continued



test values: $x = \frac{\pi}{18}$ $\quad x = \frac{\pi}{3}$ $\quad x = \frac{2\pi}{3}$ $\quad x = \frac{17\pi}{18}$

$$f'\left(\frac{\pi}{18}\right) = 6\cos\left(3\,\frac{\pi}{18}\right) + 3 = 6\cos\left(\frac{\pi}{6}\right) + 3 = (+)$$

$$f'\left(\frac{\pi}{3}\right) = 6\cos\left(3\,\frac{\pi}{3}\right) + 3 = 6\cos(\pi) + 3 = -6 + 3 = -3 = (-)$$

$$f'\left(\frac{2\pi}{3}\right) = 6\cos\left(3\,\frac{2\pi}{3}\right) + 3 = 6\cos(2\pi) + 3 = 9 = (+)$$

$$f'\left(\frac{17\pi}{18}\right) = 6\cos\left(3\,\frac{17\pi}{18}\right) + 3 = 6\cos\left(\frac{17\pi}{6}\right) + 3$$
$$= 6\cos\left(\frac{5\pi}{6}\right) + 3$$
$$= 6\left(-\frac{\sqrt{3}}{2}\right) + 3$$
$$= -3\sqrt{3} + 3 = (-)$$

By the first Derivative test,

$f(x)$ has relative maximum when

$x = \frac{2\pi}{9}$ and $x = \frac{8\pi}{9}$