03.02.22

Lesson 19

Concavity, Inflection Points & the Second Derivative Test

Concave up

- slope of Tangent lines increase from left to right

Concave down

- slope of Tangent lines decrease from left to right

Concavity of a function: Suppose $f''(x)$ exists on an open interval I

(1) If $f''(x) > 0$ for all x in I,

then $f(x)$ is concave up.

(2) If $f''(x) < 0$ for all x in I,

then $f(x)$ is concave down.

Note: The sign of the derivative tells us if a function is increasing or decreasing. $f''(x)$ is the derivative of $f'(x)$, so when $f''(x) > 0$ $f'(x)$ is increasing and when $f''(x) < 0$ $f'(x)$ is decreasing.

useful way to remember

$$\overset{+\,+}{\bigcup} \qquad\qquad\qquad \overset{-\,-}{\bigcap}$$

$f''(x) > 0$ $\qquad\qquad$ $f''(x) < 0$

concave up $\qquad\qquad$ Concave down

Example 1 $\quad$ Let $f(x) = 3x^2 + 6x + 9$

Find the intervals where $f(x)$ is concave up and concave down.

Solution

Find where $f''(x) = 0$, test the sign of $f''(x)$ in the intervals given by the x values where $f''(x) = 0$.

$$f(x) = 3x^2 + 6x + 9$$

$$f'(x) = 6x + 6$$

$$f''(x) = 6 > 0$$

$f(x)$ is concave up on $(-\infty, \infty)$

and concave down no where.

Inflection Points

(1) $f''(x) = 0$ or $f''(x)$ DNE, but $f(x)$ exists

and

(2) the concavity changes at these points

Example 2 Find where $f(x) = 4x^3 + 2x^2 + 5$ is concave up, concave down and identify any inflection points.

Solution: Find where $f''(x) = 0$ and test if the sign of $f''(x)$ changes.

$$f(x) = 4x^3 + 2x^2 + 5$$

$$f'(x) = 12x^2 + 4x$$

$$f''(x) = 24x + 4 \overset{\text{set}}{=} 0$$

$$x = -\frac{4}{24} = -\frac{1}{6}$$

$- -$ $\cap$ | $++$ $\cup$

$-\infty$ $f''(x) < 0$ $-1/6$ $f''(x) > 0$ ∞

Test values: $x = -1$, $x = 0$

$f''(x)$

$$f''(-1) = -24 + 4 = (-)$$

$$f''(0) = 4 = (+)$$

$f(x)$ is concave up

$(-\frac{1}{6}, \infty)$

concave down

$(-\infty, -\frac{1}{6})$

Inflection point at $x = -\frac{1}{6}$

or as an ordered pair

$$\left(-\frac{1}{6}, f\left(-\frac{1}{6}\right)\right) = \left(-\frac{1}{6}, \frac{1088}{216}\right)$$

Note: - Use sign of $f'(x)$ to find where $f(x)$ is increasing / decreasing

- Use sign of $f''(x)$ to find where $f(x)$ is concave up / down.

Example 2

(b) Find where $f(x) = 4x^3 + 2x^2 + 5$ is concave up and decreasing

- we already found where $f(x)$ is concave up
- Now we need to go back to $f'(x)$ and find where $f(x)$ is increasing

$$f'(x) = 12x^2 + 4x = 4x(3x+1) = 0$$

$$x = 0, \quad x = -\frac{1}{3}$$

increasing | decreasing | increasing

$f'(x) > 0$ | $f'(x) < 0$ | $f'(x) > 0$

$-\frac{1}{3}$ 0

Test values $f'(x)$: $x = -1$, $x = -\frac{1}{4}$, $x = 1$

$$f'(-1) = 4(-1)(3(-1)+1) = (-)(-) = (+)$$

$f(x)$ is increasing on $(-\infty, -\frac{1}{3})$ and $(0, \infty)$

$f(x)$ is decreasing on $(-\frac{1}{3}, 0)$

$$f'(-\tfrac{1}{4}) = 4(-\tfrac{1}{4})(3(-\tfrac{1}{4})+1) = (-)(+) = (-)$$

$$f'(1) = 4(1)(3(1)+1) = (+)$$

Example 2

(b) ... Continued

$(-\frac{1}{6}, \infty)$ concave up

$(-1/3, 0)$ decreasing

Putting the sign chart for $f'(x)$ and $f''(x)$ together we can answer the question

$f(x)$ is concave up and decreasing on the interval $(-\frac{1}{3}, 0)$

Example 3 Let $f(x) = 3\ln(x^2+2)$

Find where $f(x)$ is concave up, concave down and the inflection points.

Solution Find where $f''(x) = 0$ and test the sign on $f''(x)$ on the intervals.

$$f(x) = 3\ln(x^2+2)$$

$$f'(x) = 3\left(\frac{2x}{x^2+2}\right) = \frac{6x}{x^2+2}$$

$$f''(x) = \frac{6(x^2+2) - (6x)(2x)}{(x^2+2)^2} = \frac{6x^2+12-12x^2}{(x^2+2)^2}$$

$$= \frac{12-6x^2}{(x^2+2)^2} = 0$$

Example 3 ...cont.

$f''(x)=0$ when $12-6x^2=0 \Rightarrow 6x^2=12$

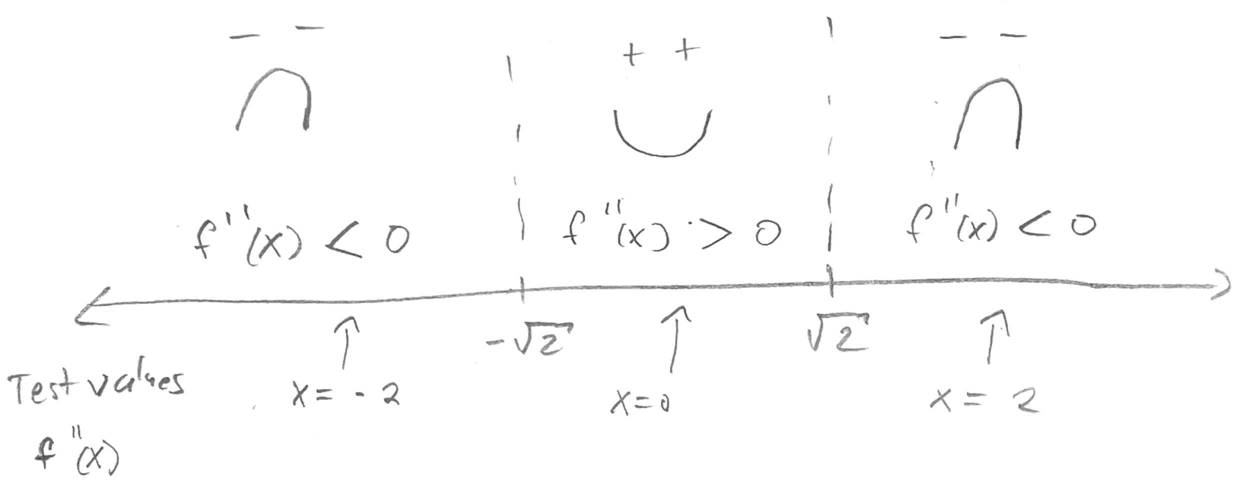
$x^2=\frac{12}{6}=2$

$x=\pm\sqrt{2}$

$f''(x)$ DNE? Nope,

$x^2+2=0 \Rightarrow x^2=-2$

has no real valued solutions



$$f''(-2)=\frac{12-6(-2)^2}{((-2)^2+2)^2}=\frac{(-)}{(+)}=(-)$$

$$f''(0)=\frac{12-6(0)}{(0+2)^2}=\frac{(+)}{(+)}=(+)$$

$$f''(2)=\frac{12-6(2)^2}{(0+2)}=\frac{(-)}{(+)}=(-)$$

inflection points

$(-\sqrt{2}, f(-\sqrt{2}))=(-\sqrt{2}, 3\ln(4))$

$(\sqrt{2}, f(\sqrt{2}))=(\sqrt{2}, 3\ln(4))$

$f(x)$ is concave up on $(-\sqrt{2}, \sqrt{2})$

concave down on $(-\infty, -\sqrt{2})$ $(\sqrt{2}, \infty)$

inflection points at $x=-\sqrt{2}$ and $x=\sqrt{2}$

Example 3

(6) Find where $f(x) = 3\ln(x^2+2)$ is concave up and increasing.

$$f'(x) = \frac{6x}{x^2+2} = 0 \quad \text{when } x = 0 \text{ only}$$

- Note $f(x)$ is defined everywhere

decreasing | increasing

$f'(x) < 0$ | $f'(x) > 0$

0

test values $f'(x)$: $x=-1$, $x=1$

$$f'(-1) = \frac{-6}{(+)} = (-) \qquad f'(1) = \frac{6}{(+)} = (+)$$

$f(x)$ is concave up and increasing on the interval $(0, \sqrt{2})$.

$(-\sqrt{2}, \sqrt{2})$ concave up

$(0, \infty)$ increasing

The Second Derivative Test

(Another way to find relative Maximums and relative minimums)

Let $f(x)$ be a function such that $f'(c) = 0$ and the second derivative of $f(x)$ exists on an open interval containing c

1. If $f''(c) > 0$, then $f(x)$ has a relative minimum $f(c)$ at $x=c$
2. If $f''(c) < 0$, then $f(x)$ has a relative maximum $f(c)$ at $x=c$

* In other cases the second Derivative Test does not apply or fails.

If that happens, you have to use the First Derivative Test from last time (L. 18)

Example 4 Let $f(x) = \frac{1}{2}x^4 - x^3 + 7$

Use the Second Derivative Test (if applicable) to find relative extrema.

Solution

$f'(x) = 2x^3 - 3x^2 = 0$ Critical Numbers

$x^2(2x-3) = 0$; $x = \frac{3}{2}$, $x = 0$

$f''(x) = 6x^2 - 6x$

Apply 2nd Derivative Test

$f''\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right)$

$= 6\left(\frac{9}{4}\right) - 9 = \frac{27}{2} - 9 > 0$

- Since $f''\left(\frac{3}{2}\right) > 0$, $x = \frac{3}{2}$ is relative minimum for $f(x)$.

- $f''(0) = 0$ So 2nd Der. Test does not apply

- Have to use 1st Der. Test to determine what happens at $x = 0$.

$f'(x) < 0$ 0 $f'(x) < 0$

Test Values $x = -1$ $x = 1$

$f'(x)$ $f'(-1) = (-1)^2(2(-1)-3) = (-)$

$f'(1) = (1)(2(1)-3) = (-)$

Example 4 ... cont.

By the first Derv. Test, $x=0$ is not a rel. min or rel. max

Notice The Second Derv. Test was faster for $x=\frac{3}{2}$. However, it did not work for $x=0$.

Example 5 Let $f(x) = 5 + 2x - 5x^3$

Use the 2nd Der. Test to find the rel. extrema for $f(x)$.

$f'(x) = 2 - 15x^2 = 0 \qquad 2 = 15x^2, \quad x = \pm\sqrt{\frac{2}{15}}$

$f''(x) = -30x$

$\left(\frac{2}{\sqrt{15}}, f\left(\frac{2}{\sqrt{15}}\right)\right) = \left(\frac{2}{\sqrt{15}}, 5.486\right)$

$f''\left(\frac{2}{\sqrt{15}}\right) = -30\left(\frac{2}{\sqrt{15}}\right) < 0$ rel. max by 2nd Der. Test

$f''\left(\frac{-2}{\sqrt{15}}\right) = -30\left(\frac{-2}{\sqrt{15}}\right) = (+) > 0$ rel. min by 2nd Der. Test

$\left(\frac{-2}{\sqrt{15}}, f\left(\frac{-2}{\sqrt{15}}\right)\right) = \left(\frac{-2}{\sqrt{15}}, 4.513\right)$