03.09.22

Lesson 22

Limits at Infinity

Recall we started the course by looking at limits like:

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

- The definition of the derivative also comes from taking a limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Now we will look at limits as x goes to infinity
  - used to find Horizontal and slant asymptotes for graphing by hand.
  - Very important for describing natural processes in nature and industry
- we will consider $x \to \infty$ and $x \to -\infty$

Example 1 Find the limits

(a) $\lim_{x \to \infty} \frac{6}{x} = 0$

(b) $\lim_{x \to -\infty} 3 + \frac{2}{x} = 3$

(c) $\lim_{x \to \infty} \frac{x}{7} = \infty$

- Why as x gets bigger. $\frac{6}{x}$ gets smaller

$\frac{6}{1000} = .0006$, so as $x \to \infty$ $\frac{6}{x} \to 0$

- Same thing happens to $\frac{2}{x}$ as $x \to -\infty$

$\frac{2}{-1000} = -.0002$

Notice the constant in the numerator does not effect the value for large x

- For (c). $\frac{x}{7}$ just keeps growing as $x \to \infty$

so $\lim_{x \to \infty} \frac{x}{7} = \infty$.

Example 2 Find the limit

$$\lim_{x \to \infty} \frac{3x^2 + 7}{5x^2 - 11}$$

Now there is competing x in the numerator and denominator. So what happens?

- We can use a table like we've done before

x	10	100	1000
$f(x)$	$\frac{3(100)+7}{5(100)-11}$	$\frac{3(10000)+7}{5(10000)-11}$	$\frac{3(1000)^2+7}{5(1000)^2-11}$
$f(x)$			.5997

- The table suggests the limit is .6 or $\frac{3}{5}$, which is correct

- However, there is an easier way. notice for $x = 1000$ the $+7$ and -11 contributed very little to the function

We could have used $\frac{3x^2}{5x^2}$ to determine the limit. Notice

$$\frac{3x^2}{5x^2} = \frac{3}{5}$$

and $\lim_{x \to \infty} \frac{3x^2+7}{5x^2-11} = \frac{3}{5}$

<u>General rule</u> The limit of a rational function as x goes to positive or negative infinity is determined by the leading terms of the numerator and the denominator.

- A leading term is the highest power of x

<u>Example 3</u> Use the general rule to find the limit

(a) $\lim_{x \to \infty} \frac{6x^3+4}{7x^2+6} \overset{\text{rule}}{=} \lim_{x \to \infty} \frac{6x^3}{7x^2} = \lim_{x \to \infty} \frac{6x}{7} = \infty$

$\lim_{x \to -\infty} \frac{6x^3+4}{7x^2+6} \overset{\text{rule}}{=} \lim_{x \to -\infty} \frac{6x}{7} = -\infty$

Example 3

rule

(b) $\lim_{x \to \infty} \dfrac{7 + 13x + 11x^2 - 6x^4}{12 + 5x^2} = \lim_{x \to \infty} \dfrac{-6x^4}{5x^2} = -\infty$

(c) $\lim_{x \to -\infty} \dfrac{x - 3x^3 + 7x^2 - 6}{2x + 3x^3 - 11} = \lim_{x \to -\infty} \dfrac{-3x^3}{3x^3} = \lim_{x \to -\infty} -\dfrac{3}{3}$

$= -1$

Horizontal Asymptote The line $y = L$ if

$\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

Recall: **Veritical Asymptote** is where

$\lim_{x \to c} f(x) = \infty, -\infty,$ or DNE

c is a fixed number (not $\pm\infty$)

or more simply, the "division by zero issues"

like $\dfrac{7}{0}$ for $f(x) = \dfrac{7}{x}$.

So $x = 0$ is a V.A.

Example 4 Find the Vertical and Horizontal Asymptotes.

(a) $f(x) = \dfrac{3}{x-7}$

V.A. $x=7$

H.A. $y=0$

Notice the different variables. One is a line, the other is a point.

$$\lim_{x\to\infty} \frac{3}{x-7} = 0$$

(b) $f(x) = \dfrac{4x^2}{x^2-25} = \dfrac{4x^2}{(x-5)(x+5)}$

V.A. at $x=5$ and $x=-5$

$$\lim_{x\to\infty} \frac{4x^2}{x^2-25} = 4$$ so H.A is $y=4$

Slant Asymptote The function approaches a line $y=ax+b$ as x goes to positive or negative infinity

- We find Slant Asymptotes using long division
- A function only has a Slant Asymptote when the polynomial in the numerator is

exactly one degree higher than the polynomial in the denominator

Example 5 Find the Slant Asymptotes of

$$f(x) = \frac{x^2+7x+12}{x-4}$$

$$\begin{array}{r|l} & x+11 \\ \hline x-4 & x^2+7x+12 \\ & -(x^2-4x) \\ \hline & 11x+12 \\ & -(11x-44) \\ \hline & 56 \end{array}$$

$$f(x) = \underbrace{x+11}_{\text{slant asymptote}} + \frac{56}{x-4}$$

$y = x+11$ is the slant asymptote

Note: $$f(x) = \frac{x^2+7x+12}{x-4} = x+11 + \underbrace{\frac{56}{x-4}}_{\text{shrinks to zero}}$$

The function looks like $y = x+11$ for large values of x.

Example 6 Find the Vertical Asymptotes, Horizontal Asymptotes and Slant Asymptotes.

$$f(x) = \frac{4x^3 + 5x^2 - 3x + 15}{x^2 + 2}$$

V.A. none $x^2 + 2$ never equals zero

H.A. none $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \frac{4x^3}{x^2} = \infty$

$$\begin{array}{r} 4x + 5 \\ x^2+2 \,\overline{)\, 4x^3 + 5x^2 - 3x + 15} \\ -(4x^3 \qquad + 8x) \\ \hline 5x^2 - 11x + 15 \\ -(5x^2 \qquad + 10) \\ \hline -11x + 5 \end{array}$$

← slant Asymptote ($4x + 5$)

← remainder ($-11x + 5$)

$$f(x) = 4x + 5 + \frac{-11x + 5}{x^2 + 2}$$

S.A. $y = 4x + 5$.

Example 7 The following do not have slant asymptotes because the degree of the numerator is not exactly one degree higher than the denominator

$$f(x) = \frac{7x^3+6}{3x-1} \quad \begin{array}{l} \leftarrow \text{deg } 3 \\ \leftarrow \text{deg } 1 \end{array}, \qquad f(x) = \frac{5x^2+x-3}{3x^4-7+x} \quad \begin{array}{l} \leftarrow \text{deg. } 2 \\ \leftarrow \text{deg } 4 \end{array}$$

$$f(x) = \frac{3x^5+x^3-7x+3}{2x^2+3x-1} \quad \begin{array}{l} \leftarrow \text{degree } 5 \\ \leftarrow \text{degree } 2 \end{array}$$