03.21.22

Lesson 23

Summary of Curve sketching

Today we put together

- x & y intercepts
- Increasing / Decreasing intervals
- concavity / Inflection points
- Horizontal, Vertical and Slant Asymptotes

To sketch an accurate graph of a function

Example 1 sketch a graph for

$$f(x) = \frac{x^2 - 6x + 12}{x - 3}$$

by finding the following information.

x-intercept: (crosses x-axis)

$$f(x) = \frac{x^2 - 6x + 12}{x - 3} = 0$$

only happens when $x^2 - 6x + 12 = 0$

(quadratic formula)

$$x = \frac{-(-6) \pm \sqrt{36 - 4(1)(12)}}{2(1)} \longleftarrow \sqrt{-12}$$

no real solutions so no x-intercepts

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Example 1 ... continued

y-intercept: (crosses y axis) When $x=0$

$$f(0) = \frac{12}{-3} = -4$$

$\boxed{(0,-4)}$

Vertical asymptote: $x = 3$

(Domain or Division by zero issues)

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2-6x+12}{x-3} = \lim_{x \to \infty} \frac{x^2}{x} = \infty$$

$$\lim_{x \to -\infty} \frac{x^2-6x+12}{x-3} = \lim_{x \to -\infty} \frac{x^2}{x} = -\infty$$

so No Horizontal asymptotes

Slant asymptotes $f(x) = \dfrac{x^2-6x+12}{x-3}$

because the top is one degree higher than the bottom, we have a slant asymptote.

Example 1 ... continued

To find the slant asymptote, we use polynomial long division.

$$\begin{array}{r l}
 & x-3 \quad \leftarrow \text{slant asymptote} \\
x-3 \,\big) & x^2-6x+12 \\
 & -(x^2-3x) \\
\hline
 & -3x+12 \\
 & -(-3x+9) \\
\hline
 & 3 \quad \leftarrow \text{remainder}
\end{array}$$

$$f(x) = \frac{x^2-6x+12}{x-3} = x-3 + \underbrace{\frac{3}{x-3}}_{\text{remainder} \to 0 \text{ as } x \to \pm\infty}$$

$\boxed{y = x-3}$ is the slant asymptote.

Increasing / Decreasing

$$f'(x) = \frac{(2x-6)(x-3) - (x^2-6x+12)(1)}{(x-3)^2}$$

$$= \frac{2x^2-12x+18-x^2+6x-12}{(x-3)^2} = \frac{x^2-6x+6}{(x-3)^2}$$

... continued

Set $f'(x)=0$ to find Critical Numbers

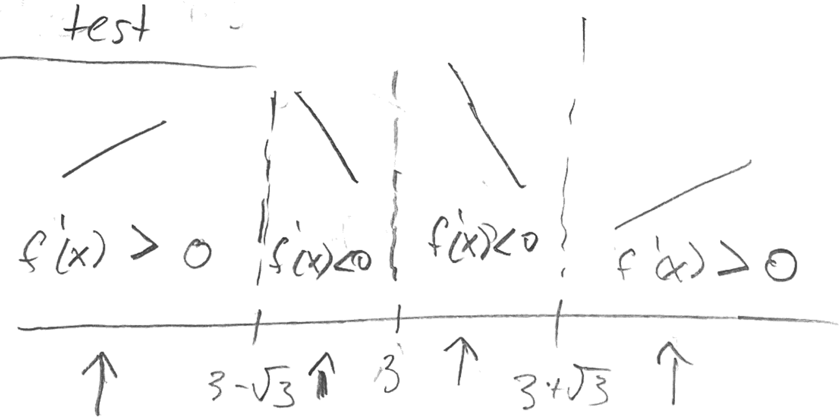
$$f'(x) = \frac{x^2-6x+6}{(x-3)^2} = 0 \quad , \text{ only when}$$

$$x^2-6x+6=0$$

$$x = \frac{-(-6) \pm \sqrt{36-4(1)(6)}}{2(1)} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2}$$
$$= 3+\sqrt{3}$$
$$\& \ 3-\sqrt{3}$$

C.N. $x = 3+\sqrt{3}$ and $x = 3-\sqrt{3}$. We also add $x=3$ to the number line because $f(3)$ DNE. (Note: $x=3$ is not a C.N.)

Number line test



Test Values for $f'(x)$: $x=0$ $\quad x=2$ $\quad x=4$ $\quad x=5$

$$f'(0) = \frac{6}{(-3)^2} > 0 \ , \qquad f'(4) = \frac{16-24+6}{(4-3)^2} = -2 < 0$$

$$f'(5) = \frac{25-30+6}{(5-3)^2} = \frac{1}{(2)^2} > 0 \qquad f'(2) = \frac{4-12+6}{(2-3)^2} < 0$$

... Continued,

rel. max at $x = 3-\sqrt{3}$

rel. min at $x = 3+\sqrt{3}$

increasing $(-\infty, 3-\sqrt{3})$ and $(3+\sqrt{3}, \infty)$

Decreasing $(3-\sqrt{3}, 3)$ and $(3, 3+\sqrt{3})$

Concavity (Need $f''(x)$)

$$f''(x) = \frac{(x-3)^2(2x-6) - (x^2-6x+6)\,2(x-3)}{(x-3)^4}$$

$$= \frac{(x-3)(2x-6) - 2(x^2-6x+6)}{(x-3)^3}$$

$$= \frac{2x^2-12x+18-2x^2+12x-12}{(x-3)^3}$$

$$= \frac{6}{(x-3)^3}$$

The numerator is never zero, but the concavity can change when $f''(x)$ DNE. at $x=3$

	$- -$		$++$
	$f''(x) < 0$		$f''(x) > 0$
Test Values for $f''(x)$	$x=0$	$x=3$	$x=4$

$$f''(0) = \frac{6}{(-3)^3} < 0$$

$$f''(4) = \frac{6}{1^3} > 0$$

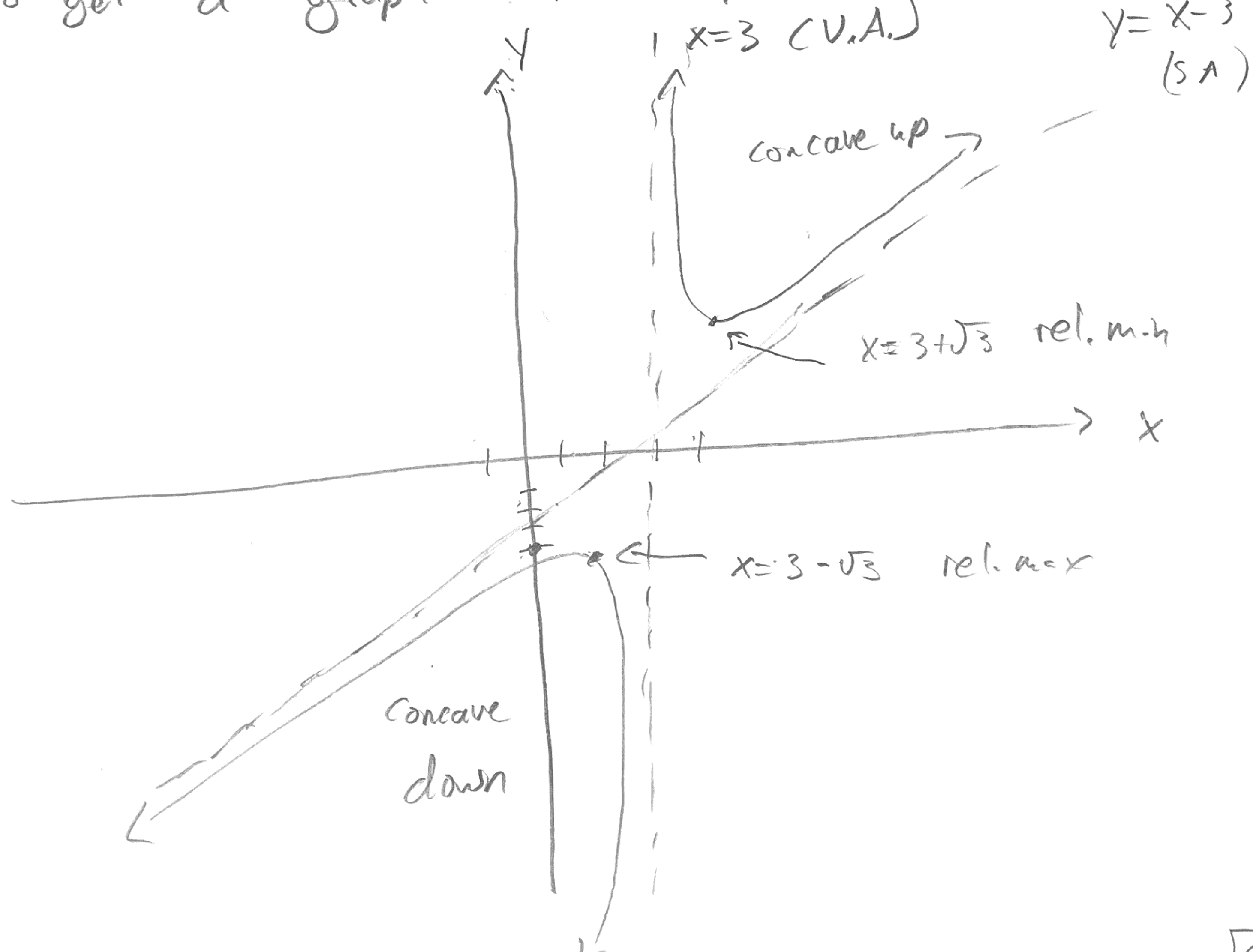
... continued !

Concave up : $(3, \infty)$

concave down! $(-\infty, 3)$

- since $f(3)$ DNE, $f(x)$ has <u>no</u> inflection points.

We can finally put all the info together to get a graph of $f(x)$.



Note when drawing the graph it is easies to add the intercepts and asymptotes first.

Example 2 $f(x) = \dfrac{x-5}{x+2}$

Sketch a graph by finding the following info.

x-int: $x = 5$; $(5, 0)$

y-int: $f(0) = \dfrac{-5}{2}$; $\left(0, -\frac{5}{2}\right)$

V.A.: $x = -2$

H.A.: $y = 1$

$$\lim_{x \to \infty} \frac{x-5}{x+2} = \lim_{x \to \infty} \frac{x}{x} = 1$$

$$\lim_{x \to -\infty} \frac{x-5}{x+2} = \lim_{x \to -\infty} \frac{-x}{-x} = 1$$

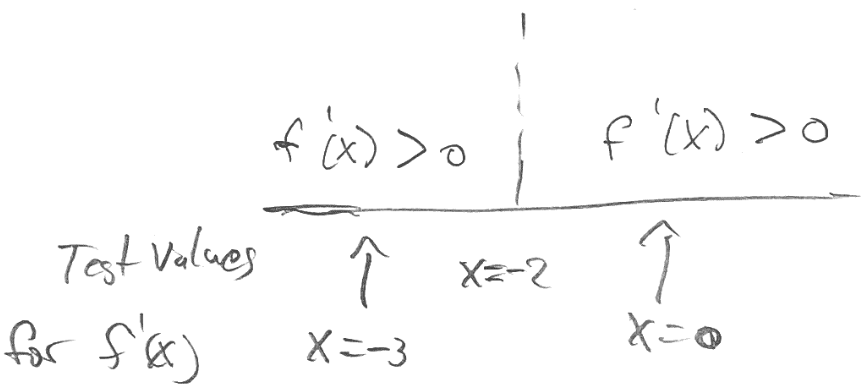
S.A.: None because the degree of the top and bottom match.

Example 2 ... cont.

increasing / decreasing

$$f'(x) = \frac{(x+2)(1) - (x-5)(1)}{(x+2)^2} = \frac{7}{(x+2)^2} \neq 0$$

There are no C.N. because $f'(x)$ never equals 0, and $f(x)$ DNE at $x=-2$. But we still need to check if $f(x)$ changes from increasing / Decreasing at $x=-2$



$$f'(0) = \frac{7}{(2)^2} > 0$$

$$f'(-3) = \frac{7}{(-3+2)^2} > 0$$

So $f(x)$ is increasing on $(-\infty,-2) \cup (-2,\infty)$ and decreasing nowhere.

Example 2 ...cont.

Concavity

$$f''(x) = 7(-2)(x+2)^{-3} = \frac{-14}{(x+2)^3}$$

No inflection points because the numerator is never 0 but we need to check if concavity changes at $x=-2$ because $f''(-2)$ DNE, and $f(-2)$ DNE.

$++$ $\qquad\qquad$ $--$

$f''(x) > 0$ $\qquad$ $f''(x) < 0$

Test values for $f''(x)$ $\qquad$ $x=-3$ $\qquad$ $x=-2$ $\qquad$ $x=0$

$$f''(-3) = \frac{-14}{(-3+2)^3} = \frac{-14}{(-1)^3} = \frac{-14}{-1} = 14 > 0$$

$$f(0) = \frac{-14}{2^3} < 0$$

concave up: $(-\infty, -2)$

concave down: $(-2, \infty)$

Example 2 ... cont.