3.23.22

Lesson 24

Optimization Part 1

A big application of calculus is finding things like maxmum profit (L.26) for a company or minimizing construction material cost.

These are called <u>optimization</u> problems.

- The function you find the maximum or minimum for is called the <u>objective function</u>.
- When there are constraints on the equation there is a second equation called the <u>constraint equation</u>.
- The general approach is to make sure the objective function is an equation of one variable by plugging in the constraint equation and then using what we've learned to find the absolute maximum or absolute minimum.

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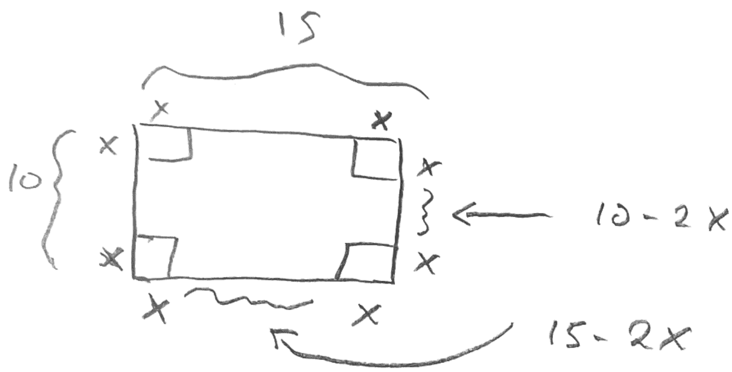
Example 1 A piece of card board is 10 inches by 15 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What is the maximum possible volume of the box? Round the answer to four decimal places.

Solution



Objective function Volume, the variable x is the edge of the square being cut out

$$V = \underset{\text{height}}{x}\,(\underset{\text{width}}{10-2x})(\underset{\text{length}}{15-2x})$$

$$= x(150 - 30x - 20x + 4x^2)$$

$$= 4x^3 - 50x^2 + 150x$$

Domain Since x is the side length of the square being cut out it must be greater than 0, but it must also be less than 5, otherwise we couldn't make a box.

Example 1 ... continued

Therefore, the domain of V is $(0, 5)$

Now we need to take the derivative of V to find the absolute maximum on $(0, 5)$.

$$V = 4x^3 - 50x^2 + 150x$$

$$V' = 12x^2 - 100x + 150 = 0$$

$$x = \frac{-(-100) \pm \sqrt{100^2 - 12(150)}}{2(12)}$$

Not in interval

$x \approx 1.962$ and $x \approx 6.371$

so $x \approx 1.962$ is the only critical number

- We have to check is x is a max or min by using either the first or second derivative test.

Second derivative Test

$$V'' = 24x - 100$$

$$V''(1.962) = 24(1.962) - 100 < 0$$

So $x = 1.962$ is a rel max and the abs max b/c no other C.N or interval end points.

Example 1 ...continued.

So we now know $x = 1.962$ maximizes the Volume function but we need the actual maximum volume. So plug $x = 1.962$ back into the original equation

$$V(1.962) = 132.0382$$

is the maximum possible volume of the box.

Example 2 A farmer is trying to fence in a rectangular field with 1200 feet of fencing. What is the maximum possible Area the field can be?

Solution

The farmer wants to maximize area so the objective function is

$$A = XY$$

The 1200 feet of fencing will be the perimeter of the field so the constraint equation is

$$2X + 2Y = 1200$$

Solve for x or y and plug into A. We can't optimize A until it is an equation of one variable.

$$2X = 1200 - 2Y$$

$$X = 600 - Y$$

Example 2 ... continued

plugging ~~x~~ into A we get

$$A = (600 - y)\,y$$
$$A = 600y - y^2$$

The Domain of A is $(0, 600)$ because there are two sides of length y of the field, it must be positive and it can't be longer than the 2x600 feet of fencing available.

- Taking the derivative to optimize A we get

$$A' = 600 - 2y = 0$$
$$600 = 2y$$
$$y = 300$$

A has one critical number and the second derivative tells us $y = 300$ is the maximum on the interval $(0, 600)$

2nd der. test: $A'' = -2 < 0 \Rightarrow y = 300$ is the abs. max.

Example 2 ... continued

Plugging back into $x = 600 - y$

we find the width $= 300$ ft and

length $= 300$ ft

So the maximum area is

$$A = (300)(300) = 90000 \text{ ft}^2.$$

Example 3 A family wants to build a yard against the side of their house with a 1000 ft^2 Area. What is the least amount of fencing needed if the house accounts for one side of the yard.

Solution

objective function $P = 2x + y$ (Perimeter)

Constraint Equation $1000 = xy$ (Area)

Example 3 ... continued

- solve for x or y in the constraint equation and plug into objective function

$$y = \frac{1000}{x}$$

$$P = 2x + \frac{1000}{x}$$

- For the domain we just know $x > 0$

- optimize P,

$$P' = 2 - \frac{1000}{x^2} = 0$$

B/c of domain

$$2 = \frac{1000}{x^2} \longrightarrow x = \pm\sqrt{500}$$

$x = \sqrt{500}$ is a C.N. and we use the 2nd derivative test to check if it is the abs. minimum

$$P'' = \frac{-1000}{x^3}(-2) = \frac{2000}{x^3}, \quad P''(\sqrt{5000}) = \frac{2000}{(5000)^{3/2}} > 0$$

Example 3 ...continued)

Therefore $x = \sqrt{500}$ is the abs. min and

$$P = 2(\sqrt{500}) + \frac{1000}{\sqrt{500}} \approx 89.443 \text{ ft}$$

is the minimum amount of fencing needed.