03.24.22

Lesson 25

Optimization Part 2

Today we continue our optimization problems

<u>Example 1</u> A Norman window is constructed by adjoining a semicircle to the top of a rectangular window as shown in the figure below. [If the perimeter of the Norman window is 25 ft, find the dimensions that will allow the window to admit the most light.]

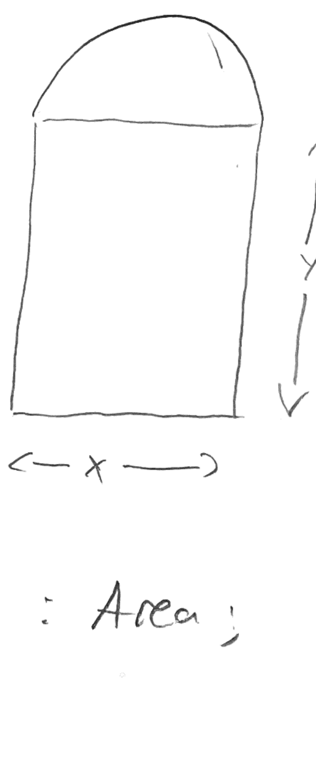
circumference of a circle

$C = 2\pi r$

semicircle

$C = \pi r$

$r = \frac{1}{2}x$



Area of circle

$A = \pi r^2$

semi-circle: $A = \frac{\pi r^2}{2}$

$r = \frac{1}{2}x$

<u>Objective function</u> : Area ; $A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

<u>Constraint Equation</u> Perimeter: $25 = 2y + x + \frac{1}{2}\pi x$

1

Example 1 ...cont

- solve for x or y in the constraint and plug into objective function

$$y = \frac{1}{2}\left(25 - x - \frac{1}{2}\pi x\right)$$

$$A = x\,\frac{1}{2}\left(25 - x - \frac{1}{2}\pi x\right) + \frac{1}{8}\pi x^2$$

$$= \frac{25}{2}x - \frac{x^2}{2} - \frac{1}{4}\pi x^2 + \frac{1}{8}\pi x^2$$

$$= \frac{25}{2}x - \frac{x^2}{2} - \frac{1}{8}\pi x^2$$

Optimize

$$A' = \frac{25}{2} - x - \frac{1}{4}\pi x = 0 \quad \text{set}$$

Find C.N.

$$x + \frac{\pi}{4}x = \frac{25}{2}$$

$$x\left(1 + \frac{\pi}{4}\right) = \frac{25}{2}$$

$$x = \frac{25}{2}\,\frac{1}{\left(1 + \frac{\pi}{4}\right)}$$

$$= \frac{25}{2 + \frac{\pi}{2}} = \frac{50}{4+\pi}$$

C.N.

Domain

comes from constraint Eq.

$x > 0$

because its a length

From the constraint

$$y = \frac{1}{2}\left(25 - x - \frac{1}{2}\pi x\right) > 0$$

$$\frac{1}{2}\left(25 - x - \frac{1}{2}\pi x\right) > 0$$

$$25 - x - \frac{1}{2}\pi x > 0$$

$$-x - \frac{1}{2}\pi x > -25$$

$$x\left(-1 - \frac{1}{2}\pi\right) > -25$$

$$x < \frac{25}{1 + \frac{1}{2}\pi}\left(\frac{2}{2}\right)$$

$$= \frac{50}{2+\pi}$$

$$\boxed{\text{Domain: } \left(0, \frac{50}{2+\pi}\right)}$$

Example 1 ... continued

$x = \frac{50}{4+\pi}$ is the only critical numbers we need to confirm it is the rel. max and abs. max

2nd Deriv. Test

$$A'' = -1 - \frac{\pi}{4} < 0$$

By 2nd Deriv. test $x = \frac{50}{4+\pi}$ is a rel. max and since it is the only C.N., also the abs min.

- We know know x will give the maximum area in $A(x)$, but we are aske for the dimensions:

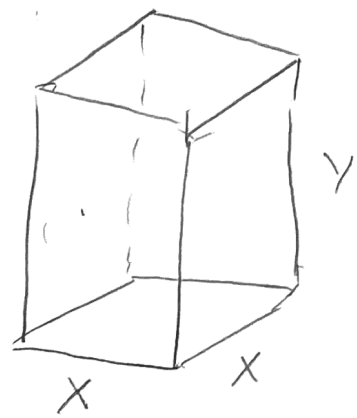
$x = \frac{50}{4+\pi}$ feet

$y = \frac{1}{2}\left(25 - \frac{50}{4+\pi} - \frac{1}{2}\pi\left(\frac{50}{4+\pi}\right)\right) = \frac{25}{4+\pi}$ feet

Example 2   A rectangular box has a square base. If the sum of the height and the perimeter of the square base is 20 in, what is the maximum possible volume?

Objective function   Volume: $V = x^2 y$

$\xrightarrow{}$ square base



Constraint Equation

$$4x + y = 20$$

- Solve for x or y and plug into objective function

$$Y = 20 - 4X$$

$$V = x^2(20-4x) = 20x^2 - 4x^3$$

Domain: $(0, 5)$ From Constraint Equation when y is close to 0)

Optimize: $V' = 40x - 12x^2 = 0$ ← set

$$4x(10-3x) = 0$$

C.N. $X = 0$ $\boxed{X = \frac{10}{3}}$

↗ Not in domain

Example 2 ...continued

- Use 2nd Deriv. Test to make sure $x = \frac{10}{3}$ is a rel. max.

2nd Deriv. Test

$$V'' = 40 - 24x$$

$$V''\left(\frac{10}{3}\right) = 40 - 24\left(\frac{10}{3}\right) = 40 - 80 < 0$$

- so $x = \frac{10}{3}$ is a rel. max. B/C it is the only C.N. in the interval is is an abs. max too.

- Plug $x = \frac{10}{3}$ back into volume to find the final answer

$$V\left(\frac{10}{3}\right) = 20\left(\frac{10}{3}\right)^2 - 4\left(\frac{10}{3}\right)^3$$

$$= \frac{2000}{27} \text{ in}^3$$

(need exact answer on HW)

is the maximum possible volume.

Example 3

A company needs to make a cylindrical can that can hold precisely 5 liters of liquid. If the entire can is to be made out of the same material, find the dimensions of the can that will minimize the cost. Round to two decimal places. Not 1 liter = 1000 cm^3

Objective function   Surface Area: $A = 2\pi r h + 2\pi r^2$

Constraint Eq   Volume: $500 = \pi r^2 h$ (b/c 5 l = 5000 cm^3)

- Solve for r or h in the constraint and plug into the objective function A.

$h = \dfrac{5000}{\pi r^2}$, $A = 2\pi r\left(\dfrac{5000}{\pi r^2}\right) + 2\pi r^2$

$A = \dfrac{10}{r} + 2\pi r^2$

Domain $(0, \infty)$ the radius r must be positive but b/c the constraint equation is volume which is multiplication there is no upper bound for r.

Optimize $A' = \dfrac{-10000}{r^2} + 4\pi r = 0$ ← Set

$4\pi r = \dfrac{10000}{r^2}$

$r^3 = \dfrac{10000}{4\pi} \longrightarrow r = \left(\dfrac{10000}{4\pi}\right)^{1/3}$

6

Example 3 ... continued,

By the second derivative test,

$$A'' = \frac{20000}{r^3} + 4\pi$$

$$A''\left(\frac{10000}{4\pi}\right)^{1/3} = \frac{20000}{\frac{10000}{4\pi}} + 4\pi > 0$$

$r = \left(\frac{10000}{4\pi}\right)^{1/3}$ is indeed the relative min and the abs. min b/c it is the only critical number.

$$h = \frac{5000}{\pi\left(\frac{10000}{4\pi}\right)^{2/3}} = 18.53 \text{ cm}$$

$$r = \left(\frac{10000}{4\pi}\right)^{1/3} = 9.267 \text{ cm}$$

Are the dimensions that will minimize the surface area and thus cost.