03/28/22

Lesson 26

Optimization Part 3/3

Example 1 A box must be constructed with volume $40\ ft^3$ using metal and wood. The metal costs $\$5/ft^2$ and the wood costs $\$2/ft^2$. If the wood is used on the sides, metal is used on the top and bottom, and the length of the base is to be 4 times the width of the base, find the dimensions of the box that will minimize the cost of construction.

Constraint Equation

Volume $V = w\,l\,h$, with $l = 4w$

Given: $V = 40\text{ ft}^3$, $l = 4w$ (length is 4 times the width)

$\to 40 = 4w^2 h$ Solve for h, $h = \frac{10}{w^2}$

Objective Function

total cost = (cost of metal top and bottom) + (cost of wood sides)

$$C = \underbrace{\$5}_{\text{cost of metal}}(\underbrace{4w^2}_{\text{top}} + \underbrace{4w^2}_{\text{bottom}}) + \underbrace{\$2}_{\text{cost of wood}}(\underbrace{2wh}_{\text{2 sides}} + \underbrace{2(4wh)}_{\text{2 sides}})$$

$$C = 40w^2 + 20wh$$

-Now plug the constraint into the objective and optimize

$$C = 40w^2 + 20w\underbrace{\left(\frac{10}{w^2}\right)}_{h}$$

Example 1 ... continued

want to minimize: $C = 40w^2 + \frac{200}{w}$

$$C' = 80w - \frac{200}{w^2} = 0$$

$$80w = \frac{200}{w^2} \rightarrow w^3 = \frac{200}{80}$$

$$\rightarrow w = \left(\frac{5}{2}\right)^{1/3}$$

Domain: $(0, \infty)$ because the dimensions on the box must be greater than 0 and the constraint is $40 = 4w^2h$.

Check $w = \left(\frac{5}{2}\right)^{1/3}$ is the minimum with 2nd Derivative test.

$$C'' = 80 - \frac{200(-2)}{w^3} = 80 + \frac{400}{w^3}$$

$$C''\left(\frac{5}{2}^{1/3}\right) = 80 + \frac{400}{\left(\frac{5}{2}\right)} > 0$$

So yes, $w = \left(\frac{5}{2}\right)^{1/3}$ is a relative min and an abs. min b/c there are no other C.N.

- So the dimensions that minimize the cost are

width $= \left(\frac{5}{2}\right)^{1/3} \approx 1.357$ length $= 4w = 4\left(\frac{5}{2}\right)^{1/3} = 5.43$

Example 1 ...cont.

and height $h = \frac{10}{\left(\frac{5}{2}\right)^{2/3}} \approx 5.43$

Example 2 A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell

$q = 2000 - 50p$ units. Each unit costs 10 dollars to make

(1) what price, p, should the company charge to maximize their revenue?

$$\text{Revenue} = (\text{price})(\text{quantity})$$
$$= p(2000 - 50p)$$
$$R = 2000p - 50p^2$$

Domain for revenue is $[0, \infty)$ since the price cannot be negative.

- Derive R to maximize

$$R' = 2000 - 100p = 0$$
$$p = \frac{2000}{100} = 20 \quad \text{C.N.}$$

Example 2 ...continued,

Now we need to check $p = 20$ is a rel max with 2nd deriv. test,

$R'' = -100 < 0$ is always negative so $p = 20$ is rel max. B/c it is the only C.N. it is also the abs. max

Final Answer: The company needs to charge $p = \$20$ to maximize revenue.

(2) What price, p, should the company charge to maximize their profits?

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

unit cost × quantity

$$= (2000p - 50p^2) - \overbrace{10\,q}$$

$$= 2000p - 50p^2 - 10(2000 - 50p)$$

$$= 2000p - 50p^2 - 20000 + 500p$$

$$P = 2500p - 50p^2 - 2000$$

Domain again $[0, \infty)$ b/c price must be positive

Example 2 (2) continued...

Derive profit $\mathcal{P}$ to optimize

$$\mathcal{P} = 2500p - 50p^2 - 2000$$

$$\mathcal{P}' = 2500 - 100p = 0$$

$$p = \frac{2500}{100} = 25$$

Check if rel max w/ second Derv. Test

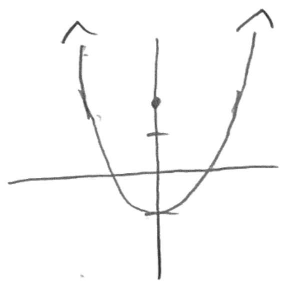
$\mathcal{P}'' = -100 < 0$

is always negative, so yes $p=25$ is a rel max. B/c $p=25$ is the only critical number, it is also abs. max.

Final Answer The company needs to charge $p = \$25$ to maximize their profits.

Example 3 Find the points on the curve $y = x^2 - 1$ closest to the point $(0, 2)$.

Solution



The distance between any two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

The objective function will be based off this.

- one point is $(0,2)$
- the other point is (x, y) from the position on the curve. So we get

$$d = \sqrt{(y-2)^2 + (x-0)^2} = \sqrt{(y-2)^2 + x^2}$$

Gives us the distance between any point on the curve and the point $(0,2)$.

- This d is tedius to take a derivative of but we can use

$$D = (y-2)^2 + x^2 \quad \text{instead}$$

Example 3 ...continued

because D is non negative and if D is minimized, then $d = \sqrt{D}$ is minimized as well.

- Now we can plug $y = x^2 - 1$ into D and take a derivative to minimize the distance.

$$D = ((x^2 - 1) - 2)^2 + x^2$$

$$= (x^2 - 3)^2 + x^2$$

$$= x^4 - 6x^2 + 9 + x^2$$

$$D = x^4 - 5x^2 + 9$$

Domain: $(-\infty, \infty)$ No restrictions on the curve.

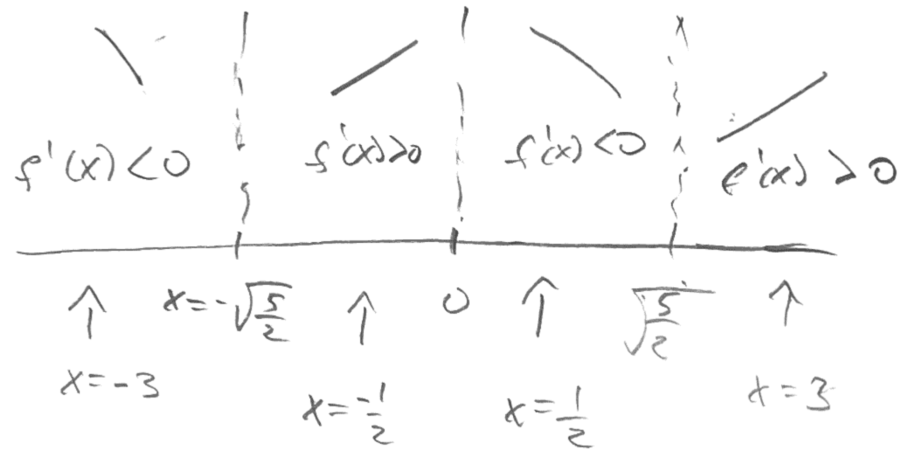
$$D' = 4x^3 - 10x = 0$$

$$2x(2x^2 - 5) = 0$$

C.N. at $x = 0$, $x = -\sqrt{\frac{5}{2}}$, $x = \sqrt{\frac{5}{2}}$

Example 3 ... continued,

Need First Derivative Test to Find rel min



Testvalues for $f'(x)$:

$$f'(-3) = 2(-3)(2(-3)^2 - 5) < 0$$

$$f'\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)\left(2\left(-\frac{1}{2}\right)^2 - 5\right) > 0$$

$$f'\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(2\left(\frac{1}{2}\right)^2 - 5\right) < 0$$

$$f'(3) = 2(3)(2(3)^2 - 5) > 0$$

rel min at $x = -\sqrt{\frac{5}{2}}$ and $x = \sqrt{\frac{5}{2}}$

The points on $y = x^2 - 1$ closest to $(0, 2)$ are $\left(-\sqrt{\frac{5}{2}}, \frac{3}{2}\right)$ and $\left(\sqrt{\frac{5}{2}}, \frac{3}{2}\right)$