03.30.22

Lesson 27

Antiderivatives and Indefinite Integration

So far in this class we have been studying differentiation and its applications like optimization.

The process has been, given a function $f(x)$ find its derivative $f'(x)$ to answer some question.

There are lots of applications, as we will learn, for the opposite process.

Given a function $f(x)$, can we find a function $F(x)$ whose derivative is $f(x)$?

That is, if: $F'(x) = f(x)$, what is $F(x)$?

- $F(x)$ is called the <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$ for all x in an interval

- Finding $F(x)$ is called <u>antidifferentiation</u>

Sometimes you can guess the antiderivative based off what we've learned about differentiation.

Example 1 Find the antiderivative and check they are correct.

$f(x) = x^2 + 6$

We can guess $F(x) = \frac{1}{3}x^3 + 6x$ by what we know about the power rule.

Then checking,

$$\frac{d}{dx}[F(x)] = \frac{1}{3}(3x^2) + 6 = x^2 + 6 = f(x)$$

So $F(x) = \frac{1}{3}x^3 + 6x$ is a antiderivative of $f(x)$.

- Notice: $G(x) = \frac{1}{3}x^3 + 6x + 7$

$H(x) = \frac{1}{3}x^3 + 6x - 2$

$J(x) = \frac{1}{3}x^3 + 6x + 10000$

Are <u>all</u> antiderivatives of $f(x) = x^2 + 6$ because the constant terms all derive to zero.

- The process of finding all antiderivatives of $f(x)$ or the general antiderivative is called

<u>indefinite integration</u>

$$\int f(x)\,dx = F(x) + C$$

where C is a constant. No matter what number C is, $F(x)+C$ is an antiderivative of $f(x)$.

- $\int$ is called the <u>integral sign</u>

- we read $\int f(x)\,dx$ as "the integral of $f(x)$, dx" or the "integral of $f(x)$".

- Integration is the reverse process of differentiation, so for every basic derivative rule we learned, we have a basic integration rule that is the opposite.

See table

- A helpful way to use the relationship between the derivative and the integral is you can always check your answer to an integral by deriving and seeing if you get back what you started with.

- The power rule for integration is:

$$\frac{d}{dx} x^n = n x^{n-1} \qquad \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

why not $n=-1$? This is our special case of $\ln x$.

$n=-1$

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad x>0 \qquad \int \frac{1}{x} \, dx = \ln |x| + C$$

(notice the absolute value signs.)

Example 2 use what we have learned about derivatives and the table of integrals to find the antiderivatives. Always write $+C$.

Note: You may have to rewrite the function before you can figure out the antiderivative

(a) $\int \frac{x^3+2}{5}\,dx$

(b) $\int \frac{2x^6+x^2}{\sqrt{x}}\,dx$

(c) $\int (2\sin x + \sec^2 x)\,dx$

(d) $\int 3\cot(x)\sin(x)\,dx$

(e) $\int \frac{2+3xe^x}{x}$

(a) $\int \frac{x^3+2}{5}\,dx = \int \left(\frac{x^3}{5} + \frac{2}{5}\right) dx$

$= \int \frac{x^3}{5}dx + \int \frac{2}{5}dx$ (can integrate around $\pm$).

$= \frac{x^4}{5}\left(\frac{1}{4}\right) + \frac{2}{5}x + C$ (power rule)

$= \frac{x^4}{20} + \frac{2}{5}x + C$ ← just need one C because the sum of two arbitrary constants is a constant.

(b) $\int \frac{2x^6+x^2}{\sqrt{x}}\,dx = \int \left(\frac{2x^6}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right) dx$

$= \int \frac{2x^6}{x^{1/2}}dx + \int \frac{x^2}{x^{1/2}}dx$

$= \int 2x^{6-1/2}dx + \int x^{2-1/2}dx$

$= \int 2x^{\frac{11}{2}}dx + \int x^{5/2}dx$

$= 2x^{\frac{11}{2}+1}\left(\frac{1}{\frac{11}{2}+1}\right) + x^{5/2+1}\frac{1}{\frac{5}{2}+1} + C$

$= 2x^{13/2}\left(\frac{2}{13}\right) + x^{7/2}\left(\frac{2}{7}\right) = \boxed{\frac{4}{13}x^{13/2} + \frac{2}{7}x^{7/2} + C}$

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(c) $\int (2\sin x + \sec^2 x)\,dx = -2\cos x + \tan x + C$

by integration table / trig rules for Derivatives.

I remember trig integrals by taking a guess and deriving to make sure it is correct.

"The derivative of $\cos x$ is $-\sin x$, so the integral of $\sin x$ must be $-\cos x$"

Indeed, $\frac{d}{dx}[-\cos x] = -(-\sin x) = \sin x$.

(d) $$\int 3\cot(x)\sin(x)\,dx = \int 3\frac{\cos x}{\sin x}(\sin x)\,dx$$

$$= \int 3\cos x\,dx$$

$$= 3\sin x + C$$

(e) $$\int \frac{2+3xe^x}{x}\,dx = \int\left(\frac{2}{x} + \frac{3xe^x}{x}\right)dx = \int \frac{2}{x}\,dx + \int 3e^x\,dx$$

$$= 2\ln|x| + 3e^x + C.$$

(rules of ln an e^x)

What about something like

$$\int x^2 \sin(3x)\,dx \quad \text{or} \quad \int x^3 e^{4x}\,dx \ ?$$

(Problem 9 & 10 HW 27)

$f(x) = x^2\sin(3x)$ and $g(x) = x^3 e^{4x}$ don't look like any basic derivative / integral rules and they are not.

The idea to find the antiderivative on the homework is to derivate the multiple choice answers and see which one gives you back $x^2\sin(3x)$ or x^3e^{4x}.

Hint both $x^2\sin(3x)$ and x^3e^{4x} are products of functions. Now the product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

has two pieces to it, but $f(x)$ and $g(x)$ only have one piece. Thus it seems reasonable that the antiderivatives of $f(x)$ and $g(x)$ should have some negative signs to cancel things out.