04.01.22

Lesson 28

Indefinite Integration with an initial condition.

- Last time we had the indefinite integral which only gives us the antiderivative up to a constant.

$$\int f(x)\,dx = F(x) + C$$

- Today we will find a particular solutions based off an initial condition.

Example 1 Given $f'(x) = 6x^2 + 3x$ and the initial condition $f(1) = 8$, find $f(x)$.

$$f(x) = \int f'(x)\,dx = \int (6x^2 + 3x)\,dx$$
$$= 2x^3 + \frac{3}{2}x^2 + C$$

$$f(1) = 2(1)^3 + \frac{3}{2}(1)^2 + C = 8$$

$C = 8 - 2 - \frac{3}{2}$, so $C = \frac{9}{2}$

Example 1 ...continued

So the particular solution based off the initial condition is

$$f(x) = 2x^3 + \frac{3}{2}x^2 + \frac{9}{2}.$$

1- A differential equation in x and y is an equation that relates x, y and the derivatives of y.

- So if we write the last example as

$$y' = 6x^2 + 3x$$

then it is a differential equation.

general solution $\quad y = 2x^3 + \frac{3}{2}x^2 + C$

- If you are given at least one y value like in example 1, called the initial condition $y(1) = 8$, then you can find the particular solution like we did.

If you are given a differential equation and an initial condition it is called an <u>initial value problem</u>.

<u>Example 2</u> Solve the initial value problem

$y'' = 5x + 3$ with $y'(1) = 2$ and $y(0) = 3$

$$y' = \int y''\,dx = \int (5x+3\,dx)$$

$$= \frac{5}{2}x^2 + 3x + C$$

$$y'(1) = \frac{5}{2}(1)^2 + 3(1) + C = 2$$

$$C = 2 - \frac{5}{2} - 3 = \frac{4}{2} - \frac{5}{2} - \frac{6}{2} = -\frac{7}{2}$$

$$y = \int y'\,dx = \int \left(\frac{5}{2}x^2 + 3x - \frac{7}{2}\right) dx$$

$$= \frac{5}{2}x^3\left(\frac{1}{3}\right) + 3x^2\left(\frac{1}{2}\right) - \frac{7}{2}x + C$$

$$y(0) = \frac{5}{6}(0)^3 + \frac{3}{2}(0)^2 - \frac{7}{2}(0) + C = 3$$

so $C = 3$ and

$$y = \frac{5}{6}x^3 + \frac{3}{2}x^2 - \frac{7}{2}x + 3$$

Example 3 Solve the initial value problem

$$y' = 3\sin x + 4 \quad \text{with} \quad y\left(\frac{\pi}{2}\right) = 1$$

$$y = \int y'\,dx = \int (3\sin x + 4)\,dx$$

$$= -3\cos x + 4x + C$$

$$y\left(\frac{\pi}{2}\right) = -3\cos\left(\frac{\pi}{2}\right) + 4\left(\frac{\pi}{2}\right) + C = 1$$

$$C = -2\pi + 1$$

$$y = -3\cos x + 4x - 2\pi + 1$$

Example 4 Given $y' = \frac{3}{x} + 2e^{x}$ with $y(1) = 6$, Find $y(e)$

$$y = \int y'\,dx = \int \left(\frac{3}{x} + 2e^{x}\right) dx$$

$$= 3\ln|x| + 2e^{x} + C$$

$$y(1) = \underbrace{3\ln(1)}_{=0} + 2e^{(1)} + C = 6$$

$$C = 6 - 2e$$

$$y = 3\ln|x| + 2e^{x} + 6 - 2e$$

$$y = \underbrace{3\ln|e|}_{=3} + 2e^{e} + 6 - 2e = 9 + 2e^{e} - 2e.$$

Example 5

A hot air balloon is rising vertically with a velocity of 8 feet per second. A ball is released from the hot air balloon at the instant when it is 400 feet above the ground. Use $a(t) = -32$ ft/sec^2 as the acceleration due to gravity.

1. How many seconds after release does the ball hit the ground?
2. What is the velocity when the ball hits the ground.

Solution Given acceleration $a(t) = -32$ ft/sec^2 we have to find height $h(t)$, where $h''(t) = a(t)$. We know $v(0) = 8$ and $h'(t) = v(t)$. We are also given $h(0) = 400$, Once we have $h(t)$, solving $h(t) = 0$ will give us the answer

$$v(t) = \int a(t)\,dt = \int -32\,dt$$

$$= -32t + C$$

$v(0) = -32(0) + C = 8$ so $C = 8$ and

$v(t) = -32t + 8$

continues...

Example 5 ... continued

$$h(t) = \int v(t)\,dt = \int (-32t + 8)\,dt$$

$$= -16t^2 + 8t + C$$

$h(0) = -16(0)^2 + 8(0) + C = 400$, so $C = 400$ and

$$h(t) = -16t^2 + 8t + 400$$

To find the time the ball hits the ground,

set $h(t) = 0$ and solve for t

$$-16t^2 + 8t + 400 = 0$$

$$-8(2t^2 - t - 50) = 0$$

$$t = \frac{-(-1) \pm \sqrt{1^2 - 4(2)(-50)}}{2(2)}$$

$$= \frac{1 \oplus \sqrt{401}}{4} \approx 5.26 \text{ seconds}$$

2. $V(5.24) = -32(5.24) + 8 = -160.20$ ft/sec.