04.08.22

# Lesson 29

# Area and Riemann Sums

Today

- introduce Summation notation
- Riemann Sums - integration as area under a curve.

Summation Notation    Write the sum using Summation Notation

Given, $1 + 4 + 9 + 16 + 25$

$= 1 + 2^2 + 3^2 + 4^2 + 25$

$= \sum_{i=1}^{5} (i)^2$    ending number

index    starting number

Evaluate    $\sum_{i=2}^{4} \sqrt{i}\,(1-i^2) = \sqrt{2}\,(1-(2)^2) + \sqrt{3}\,(1-(3)^2)$

$+ \sqrt{4}\,(1-(4)^2)$

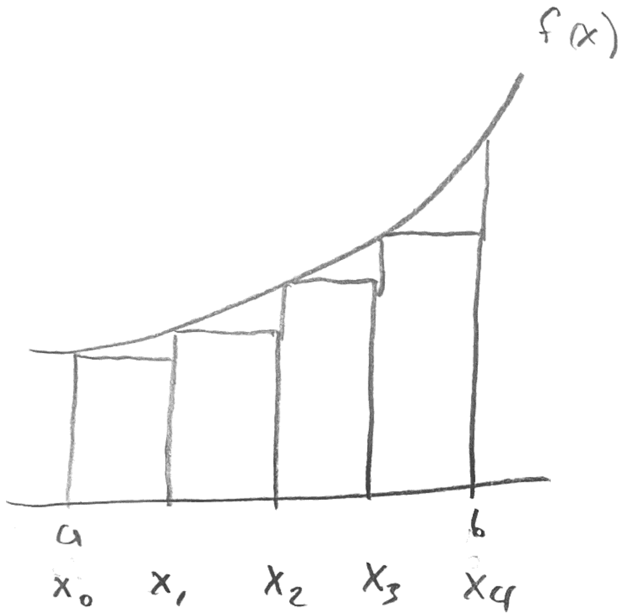
$= \sqrt{2}(-3) + \sqrt{3}\,(-8) + 2\,(-15)$

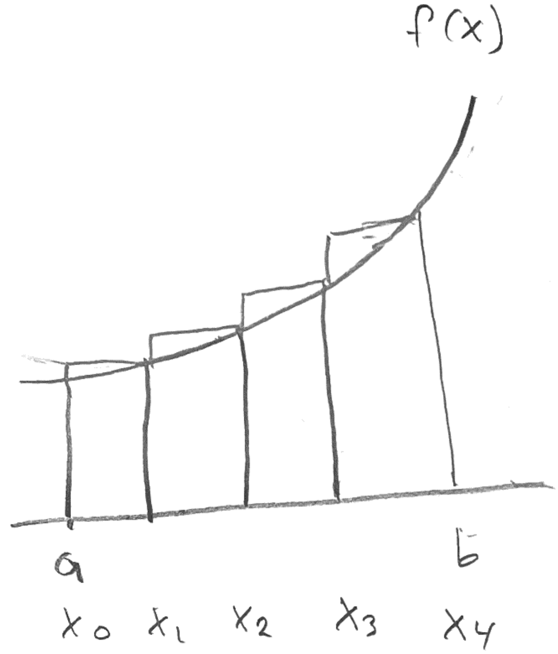
$= -48.099$

Example   Write using Sigma notation

$$(4\sqrt{0}+3)+(4\sqrt{1}+4)+\cdots+(4\sqrt{n}+n+3)$$

$$\sum_{i=0}^{n}(4\sqrt{i}+i+3)$$




Left Riemann Sum

$$L_n=\sum_{i=0}^{n-1} f(x_i)\cdot\Delta x$$

Right Riemann Sum

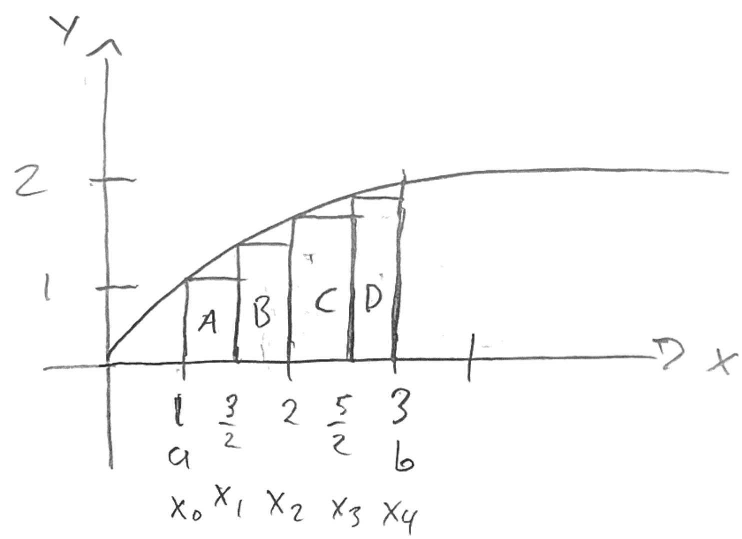
$$R_n=\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

where $x_i = a + i\cdot\Delta x$ and $\Delta x = \dfrac{b-a}{n}$

The Left and Right Riemann Sums are two different ways to estimate the signed area under a curve.


The more rectangles you have, the closer the approximation gets.

<u>Example</u> Find the Left and Right Riemann Sums with 4 rectangles to estimate the area under the curve of $y = \sqrt{x}$ on the interval of $[1, 3]$



Example.. Continued

$$\text{Area A: } \overset{\text{height}}{f(x_0)}\ \overset{\text{base}}{(x_1 - x_0)} = f(1)\left(\tfrac{3}{2} - 1\right) = \sqrt{1}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\text{Area B: } f(x_1)(x_2 - x_1) = f\left(\tfrac{3}{2}\right)\left(2 - \tfrac{3}{2}\right) = \sqrt{\tfrac{3}{2}}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}$$

$$\text{Area C: } f(x_2)(x_3 - x_2) = f(2)\left(\tfrac{5}{2} - 2\right) = \sqrt{2}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{2}$$

$$\text{Area D: } f(x_3)(x_4 - x_3) = f\left(\tfrac{5}{2}\right)\left(3 - \tfrac{5}{2}\right) = \sqrt{\tfrac{5}{2}}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\tfrac{5}{2}}$$

Left Riemann Sum = Area A + Area B + Area C + Area D

Notice width was always the same!

$$\Delta x = \frac{b - a}{\#\text{ rectangle}} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$$

$\Delta x = \frac{1}{2}$ for all 4 rectangle

What did we plug into $f(x)$?

$$x_i = a + i \cdot \Delta x$$

Check: $x_0 = 1 + 0 \cdot \Delta x = 1$, $x_1 = 1 + 1\left(\frac{1}{2}\right) = \frac{3}{2}$

<u>Example</u> ... continued

$$x_2 = 1 + 2\left(\tfrac{1}{2}\right) = 2 \quad , \quad x_3 = 1 + 3\left(\tfrac{1}{2}\right) = \tfrac{5}{2}$$

<u>Note:</u> we only used $i = 0, 1, 2, 3$

Combining all of this info and using Sigma notation we get

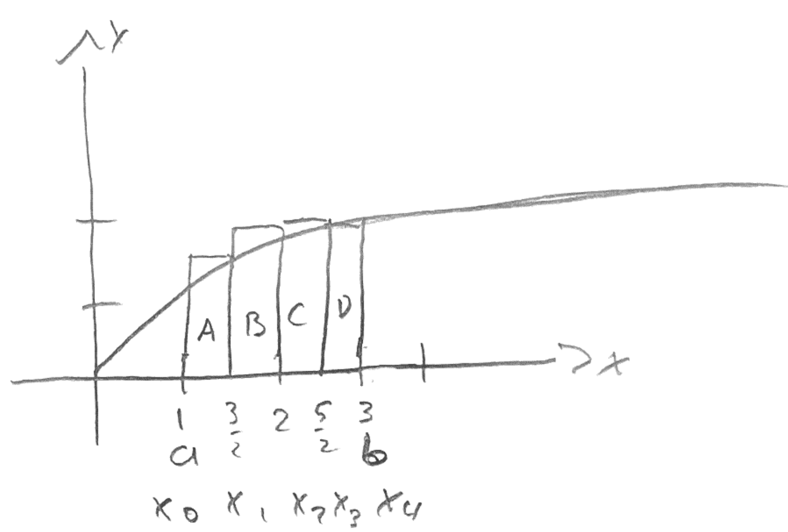
$$L_4 = \sum_{i=0}^{3} f(x_i)\cdot \Delta x = \sum_{i=0}^{3} \left(\sqrt{a + i\cdot\Delta x}\right)\Delta x$$

$$= \sum_{i=0}^{3} \sqrt{1 + \tfrac{1}{2} i}\,\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{2} \sum_{i=0}^{3} \sqrt{1 + \tfrac{1}{2} i}$$

So $L_4 = \frac{1}{2}\left(1 + \sqrt{\frac{3}{2}} + \sqrt{2} + \sqrt{\frac{5}{2}}\right) = 2.61$

Now for the Right Riemann Sum



Area A: $f(x_1)\Delta x = \sqrt{\frac{3}{2}}\left(\frac{1}{2}\right)$

Area B: $f(x_2)\Delta x = \sqrt{2}\left(\frac{1}{2}\right)$

Area C: $f(x_3)\Delta x = \sqrt{\frac{5}{2}}\left(\frac{1}{2}\right)$

Area D: $f(x_4)\Delta x = \sqrt{3}\left(\frac{1}{2}\right)$

Using Sigma notation we can write the Right Riemann Sum as

$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$= \sum_{i=1}^{4} \sqrt{a + i\cdot\Delta x}\,\Delta x$$

$$= \frac{1}{2}\sum_{i=1}^{4} \sqrt{1 + \frac{1}{2}i}$$

so

$$R_4 = \frac{1}{2}\left(\sqrt{\frac{3}{2}} + \sqrt{2} + \sqrt{\frac{5}{2}} + \sqrt{3}\right) = 2.976$$

Notice the index for $L_4$ is different than $R_4$

and $L_4 \neq R_4$

Example

Write the formula for the Left and Right Riemann Sums with 100 rectangles to estimate the (signed) area under the curve

(a) $f(x) = 3x - 1$ on the interval $[0, 25]$

$$\Delta x = \frac{25-0}{100} = \frac{1}{4}, \quad x_i = a + i \cdot \Delta x = 0 + i \cdot \frac{1}{4}$$

$$L_{100} = \sum_{i=0}^{99} f\left(\frac{i}{4}\right) \cdot \frac{1}{4} = \frac{1}{4} \sum_{i=0}^{99} \left(3\left(\frac{i}{4}\right) - 1\right) = 3612.5$$

$$R_{100} = \sum_{i=1}^{100} f\left(\frac{i}{4}\right) \cdot \frac{1}{4} = \frac{1}{4} \sum_{i=1}^{100} \left(3\left(\frac{i}{4}\right) - 1\right) = 3687.5$$

(b) $f(x) = e^{3x} + 4$ on the interval $[20, 40]$

$$\Delta x = \frac{40-20}{100} = \frac{20}{100} = \frac{1}{5}, \quad x_i = a + i \cdot \Delta x = 20 + i \cdot \frac{1}{5}$$

$$L_{100} = \sum_{i=0}^{99} f\left(20 + \frac{i}{5}\right) \cdot \frac{1}{5} = \frac{1}{5} \sum_{i=0}^{99} \left(e^{3\left(20 + \frac{i}{5}\right)} + 4\right)$$

$$R_{100} = \frac{1}{5} \sum_{i=1}^{100} \left(e^{3\left(20 + \frac{i}{5}\right)} + 4\right)$$

Example Use the Left and Right Riemann Sums with 3 rectangles to estimate the area under the curve of $y = \ln x$ on the interval $[3, 12]$

$$\Delta x = \frac{12-3}{3} = \frac{9}{3} = 3 \quad , \quad x_i = a + i \cdot \Delta x = 3 + i \cdot 3$$

$$L_3 = 3\sum_{i=0}^{2} \ln(3+3i) = 3(\ln(3) + \ln(6) + \ln(9)) = 15.263$$

$$R_3 = 3\sum_{i=1}^{3} \ln(3+3i) = 3(\ln(6) + \ln(9) + \ln(12)) = 19.422$$