04.15.22 Lesson 32

The Fundamental Theorem of Calculus

Today we connect the indefinite and definite integral.

The Fundamental Theorem of Calculus

Suppose $f(x)$ is continuous on the interval $[a,b]$. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We compute $\int_a^b f(x)\,dx$ by writing

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Use the Fundamental Theorem of Calculus on the following examples.

Example 1 Evaluate

$$\int_1^3 (8x^2+4)\,dx$$

$$= 4\int_1^3 (2x^2+1)\,dx = 4\left(2\frac{x^3}{3}+x\right)\Bigg|_1^3$$

$$= 4\left(2\frac{(3)^3}{3}+3\right) - 4\left(2\left(\frac{1}{3}\right)+1\right)$$

$$= 4\left(18+3-\frac{2}{3}-1\right)$$

$$= 4\left(20-\frac{2}{3}\right) = 4\left(\frac{60}{3}-\frac{2}{3}\right) = \frac{4(58)}{3} = \frac{232}{3}$$

Example 2 Evaluate

$$\int_0^8 \left(3e^x + \frac{\sqrt[3]{x^2}}{3}\right)dx = \int_0^8 \left(3e^x + \frac{x^{2/3}}{3}\right)dx$$

$$= \left(3e^x + \frac{x^{2/3+1}}{3}\,\frac{1}{\frac{2}{3}+1}\right)\Bigg|_0^8$$

$$= \left(3e^x + \frac{x^{5/3}}{3}\,\frac{3}{5}\right)\Bigg|_0^8$$

$$= \left(3e^8 + \frac{(8)^{5/3}}{5}\right) - (3e^0+0)$$

$$= 3e^8 + \frac{32}{5} - 3$$

Example 3 Evaluate

$$\int_{-2}^{3} \frac{x^3+2}{5}\,dx = \frac{1}{5}\int_{-2}^{3}(x^3+2)\,dx$$

$$= \frac{1}{5}\left(\frac{x^4}{4}+2x\right)\Big|_{-2}^{3}$$

$$= \frac{1}{5}\left(\frac{3^4}{4}+6\right) - \frac{1}{5}\left(\frac{(-2)^4}{4}+(2)(-2)\right)$$

$$= \frac{1}{5}\left(\frac{81}{4}+6-4+4\right)$$

$$= \frac{1}{5}\left(\frac{81}{4}+\frac{24}{4}\right) = \frac{105}{20} = \frac{21}{4}$$

Example 4 Evaluate

$$\int_{-\pi/2}^{\pi/2}(3\cos x + 4)\,dx$$

$$= (3\sin x + 4x)\Big|_{-\pi/2}^{\pi/2}$$

$$= \left(3\sin\left(\frac{\pi}{2}\right)+4\left(\frac{\pi}{2}\right)\right) - \left(3\sin\left(-\frac{\pi}{2}\right)+4\left(-\frac{\pi}{2}\right)\right)$$

$$= (3+2\pi) - (-3-2\pi)$$

$$= 6 + 4\pi$$

Example 5 Find the area of the region bounded by the graphs of

$$Y = 3x+7, \quad y=0, \quad x=3 \quad \text{and} \quad x=8$$

$$\text{Area} = \int_3^8 (3x+7)\,dx = \left(\frac{3x^2}{2} + 7x\right)\Big|_3^8$$

$$= \frac{3(64)}{2} + 7(8) - \left(\frac{3(9)}{2} + 7(3)\right)$$

$$= 3(32) + 56 - \frac{27}{2} - 21$$

$$= \frac{235}{2}$$

Example 6 Find the area enclosed by the graphs of the following equations

$$Y = 4\left(\frac{x}{2} - \frac{\sqrt{x}}{2}\right)^2, \quad Y=0, \quad x=1, \quad x=4$$

$$\text{Area} = \int_1^4 4\left(\frac{x}{2} - \frac{\sqrt{x}}{2}\right)^2 dx = \int_1^4 4\left(\frac{x^2}{4} - 2\frac{x^{3/2}}{4} + \frac{x}{4}\right) dx$$

$$= \int_1^4 (x^2 - 2x^{3/2} + x)\,dx$$

Example 6 ... Continued

$$\int_1^4 (x^2 - 2x^{3/2} + x)\,dx = \left(\frac{x^3}{3} - \frac{4x^{5/2}}{5} + \frac{x^2}{2}\right)\Bigg|_1^4$$

$$= \left(\frac{4^3}{3} - \frac{4}{5}(4)^{5/2} + \frac{(4)^2}{2}\right) - \left(\frac{1}{3} - \frac{4}{5} + \frac{1}{2}\right)$$

$$= \frac{64}{3} - \frac{128}{5} + \frac{16}{2} - \frac{1}{3} + \frac{4}{5} + \frac{1}{2}$$

$$= 21 - \frac{124}{5} + \frac{15}{2} = \frac{37}{10} = 3.7$$

Example 7 Evaluate

$$\int_0^{\pi} \sec x\,(3\sec x + 4\tan x)\,dx$$

$$= \int_0^{\pi} (3\sec^2 x + 4\sec x \tan x)\,dx$$

$$= 3\tan x + 4\sec x \Big|_0^{\pi}$$

$$= (3\tan(\pi) + 4\sec(\pi)) - (3\tan(0) + 4\sec(0))$$

$$= 3(0) + 4(-1) - 0 - 4(1)$$

$$= -8$$