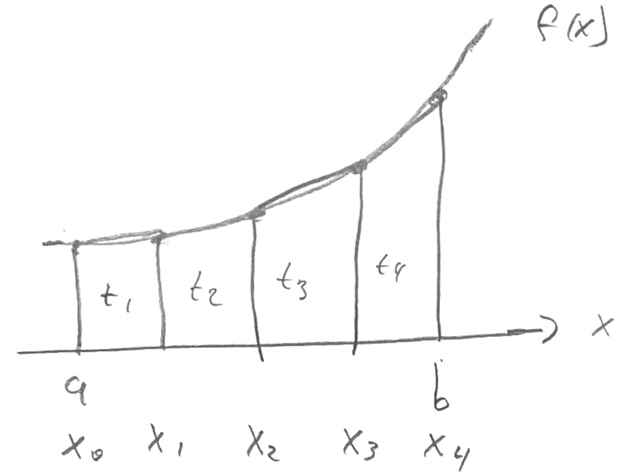
04.20.22 Lesson 34

Numerical Integration

- There are many functions we are unable to integrate $f(x) = \frac{\sin x}{x+1}$ and $f(x) = e^{x^2}$ are two examples
- In cases like these we must return to approximations
- We will use the Trapezoid Rule it is similar to the left and Right Riemann sums, but is a better approximation
- The Trapezoid Rule can be programed into a computer and calculated to be a highly accurate approximation.

TI

Suppose $f(x)$ is continuous on $[a,b]$. Then we can approximate $\int_a^b f(x)\,dx$ using trapezoids.



$$\text{Area } t_1 = \frac{1}{2}(\text{base } 1 + \text{base } 2)(\text{width})$$

$$= \frac{1}{2}\left(f(x_0) + f(x_1)\right)\cdot \Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ ← number of trapezoids and $x_i = a + i\Delta x$

$$t_2 = \frac{1}{2}\left(f(x_1) + f(x_2)\right)\cdot \Delta x$$

$$t_3 = \frac{1}{2}\left(f(x_2) + f(x_3)\right)\cdot \Delta x$$

$$t_4 = \frac{1}{2}\left(f(x_3) + f(x_4)\right)\cdot \Delta x$$

$$T_4 = \frac{1}{2}\left(f(x_0) + f(x_1)\right)\cdot \Delta x + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\cdot \Delta x + \frac{1}{2}\left(f(x_2) + f(x_3)\right)\cdot \Delta x$$

$$+ \frac{1}{2}\left(f(x_3) + f(x_4)\right)\cdot \Delta x$$

$$= \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

Notice Δx and x_i are the same as in Riemann Sums.

Example 1 Use the Trapezoid rule to approximate $\int_2^7 (2x^2+1)\,dx$ using $n=4$ and compare with the actual value of $\int_2^7 (2x^2+1)\,dx$.

Solution

$$\Delta x = \frac{7-2}{4} = \frac{5}{4} \quad ; \quad x_i = 2 + i\left(\frac{5}{4}\right)$$

$$T_4 = \frac{1}{2}\underbrace{\left(\frac{5}{4}\right)}_{\Delta x}\Big(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\Big)$$

$$x_0 = 2 \; ; \; x_1 = 2 + \frac{5}{4} = \frac{13}{4}, \quad x_2 = 2 + 2\left(\frac{5}{4}\right) = \frac{9}{2}$$

$$x_3 = 2 + 3\left(\frac{5}{4}\right) = \frac{23}{4}, \quad x_4 = 7$$

$$T_4 = \frac{5}{8}\Big(\big(2(2)^2+1\big) + 2\big(2\left(\tfrac{13}{4}\right)^2+1\big) + 2\big(2\left(\tfrac{9}{2}\right)^2+1\big) + 2\big(2\left(\tfrac{23}{4}\right)^2+1\big) + \big(2(7)^2+1\big)\Big)$$

(extra 2 b/c counted twice)

$$= \frac{5}{8}\left(9 + 44.25 + 83 + 92.25 + 99\right) \approx 230.9375$$

Example 1 ... continued

$$\int_2^7 (2x^2+1)\,dx = \frac{685}{3} \approx 228.333$$

Percent Error $\dfrac{|\text{Approx.} - \text{Exact}|}{\text{Exact}}$

$$= \frac{|230.9375 - 228.333|}{228.333} \approx 0.012$$

1.2 % Error with 4 triangles.

Gets better with more triangles.

Example 2 Use the Trapezoidal Rule to approximate $\int_4^6 \ln(x^2+3)\,dx$ using $n=3$.

Solution

$$\Delta x = \frac{6-4}{3} = \frac{2}{3} \;; \quad x_i = 4 + i\cdot\frac{2}{3}$$

$$x_0 = 4, \quad x_1 = 4 + \frac{2}{3} = \frac{14}{3}, \quad x_2 = 4 + 2\left(\frac{2}{3}\right) = \frac{16}{3}$$

$$x_3 = 4 + 3\left(\frac{2}{3}\right) = 6$$

$$T_4 = \frac{1}{2}\underbrace{\left(\frac{2}{3}\right)}_{\Delta x}\Bigg(\underbrace{\ln((4)^2+3)}_{f(x_0)} + \underbrace{2\left(\ln\left(\left(\frac{14}{3}\right)^2+3\right)\right)}_{2f(x_1)} + \underbrace{2\left(\ln\left(\left(\frac{16}{3}\right)^2+3\right)\right.}_{2f(x_2)} + \underbrace{\ln(6^2+3)}_{f(x_3)}\Bigg)$$

Example 2 ... continued

$$T_4 = \frac{1}{3}(2.944 + 6.412 + 6.896 + 3.663)$$

$$\approx 6.641$$

Example 3 Use the Trapezoid Rule to approximate

$$\int_{-0.9}^{1.5} \frac{\sec(x)}{x+2}\,dx$$ using $n=3$. Round to 4 decimal places.

Solution

$$\Delta x = \frac{1.5-(-0.9)}{3} = \frac{2.4}{3} = .8$$

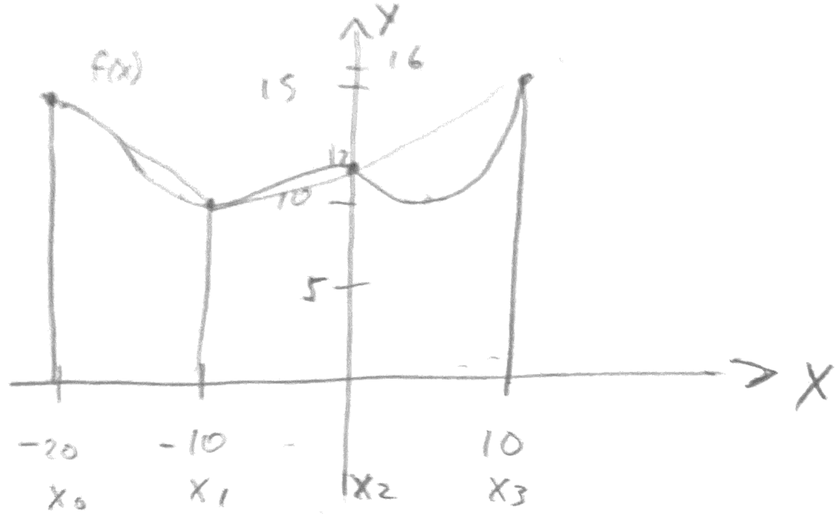
$$x_i = -0.9 + i(.8)$$

$x_0 = -0.9$, $x_1 = -0.1$, $x_2 = 0.7$, $x_3 = 1.5$

$$T_3 = \frac{1}{2}\overbrace{(1.2)}^{\Delta x}\left(\overbrace{\frac{\sec(-.9)}{-.9+2}}^{f(x_0)} + \frac{2\sec(0.3)}{.3+2} + \frac{2\sec(1.5)}{1.5+2} + \frac{\sec(2.7)}{2.7+2}\right)$$

$$\approx 1.8820$$

Example 4



Approximate the area using the Trapezoid Rule with $n=3$. $\Delta x = \frac{10-(-20)}{3} = 10$

$$T_3 = \frac{1}{2} \cdot 10 \left(f(-20) + 2f(-10) + 2f(0) + f(10) \right)$$

$$= 5 \left(15 + 2(10) + 2(12) + 15 \right)$$

$$= 5 \left(30 + 20 + 24 \right) = 5 (74) = 370$$

Notice: This approximation is not too percise with $n=3$