04.21.22

Lesson 35

Exponential Growth

Idea: How would we solve the initial value problem $\frac{dy}{dt} = 3y$ with $y(0) = 100$?

writing it another way we have

$$y'(t) = 3\,y(t)$$

what function do we know that is its own derivative? $y = e^t$, but to get the constant 3 out front, we need $y = e^{3t}$

deriving to check we see $y'(t) = 3e^{3t} = 3\,y(t)$

However, any multiple of e^{3t} also works

So a general solution is $y = Ce^{3t}$ where C is a constant. We use the initial condition $y(0) = 100$ to find C.

$$y(0) = Ce^{3(0)} = 100 \rightarrow C = 0$$

So $y = 100e^{3t}$ is the solution to the initial value problem.

Example 1 Find $y(t)$ if $\frac{dy}{dt} = 2y$ and $y(1) = 200$.

Solution The general solution is

$$y(t) = Ce^{2t}$$

plug in $t=1$ to solve for C.

$$y(1) = Ce^2 = 200 \rightarrow C = \frac{200}{e^2}$$

so $$y(t) = \frac{200}{e^2} e^{2t} = 200e^{2(t-1)}$$

Example 2 The population of a culture of bacteria, $P(t)$, where t is time in days, is growing at a rate that is proportional to the population itself and the growth rate is 0.4. The initial population is 40.

(a) What is the population after 20 days?

(b) How long does it take the population to double?

Example 2 ... continued.

Solution

- We are told "The growth rate is proportional to the population itself" that translates to

$$\frac{dP}{dt} = \underset{\substack{\uparrow \\ \text{Growth rate}}}{0.4}\, P$$

Therefore, we know the population is an exponential function.

$$P(t) = C e^{0.4(t)}$$

- Initial population is 40 translates to

$$P(0) =$$

So, $P(0) = Ce^{0.4(0)} = 0 \quad \longrightarrow \quad C = 40$

and $P(t) = 40e^{0.4t}$ is equation for the population of the bacteria where t is in days.

With this equation in hand we can answer

(a). $P(20) = 40e^{0.4(20)} = 40e^{8} \approx 119238$

$\uparrow$

what lou cappa wants

Example 2 ... continued

(6) The initial population is 40, so we want to know how many days it takes for the population to be 80.

$$P(t) = 40e^{0.4(t)} = 80$$

$$e^{0.4(t)} = \frac{80}{40} = 2$$

$$\ln\left(e^{0.4(t)}\right) = \ln(2)$$

$$0.4t = \ln 2$$

$$t = \frac{\ln 2}{0.4} \approx 1.7$$

It takes 1.7 days for the population to double.

Example 3 The rate of change of the population of a brood of cicadas is $\frac{dP}{dt} = k \cdot P$, where P is the population, t is time in years and k is the growth rate.

If $P = 30000$ when $t = 4$ and 60000 when $t = 5$, what is the population when $t = 10$?

Round to the nearest integer.

Example 3 ... continued

Solution

We know population is an exponential function with

$$P(t) = Ce^{kt}$$

- We need to find both C and k to get the exact equation for $P(t)$. Then we can find $P(10)$.

- Since we are given $P(3) = 30000$ and $P(4) = 60000$ that is enough info to find C and k.

$$P(4) = Ce^{4k} = 30000$$

$$P(5) = Ce^{5k} = 60000$$

- Solve for C in one eq: $\quad C = \dfrac{30000}{e^{4k}}$

- Plug into other eq: $\quad \dfrac{30000}{e^{4k}} e^{5k} = 60000$

Which becomes: $\quad e^{5k-4k} = e^{k} = \dfrac{60000}{30000} = 2$

$$\ln(e^{k}) = \ln 2$$

$$k = \ln 2$$

- Now that we have k, we can find C.

$$C = \frac{30000}{e^{4\ln 2}} = \frac{30000}{e^{\ln 2^4}} = \frac{30000}{16} = 1875$$

Example 3 ... continued.

Plugging c and k into the original equation we have that

$$P(t) = 1875 e^{(\ln 2) t}$$

and

$$P(10) = 1875 e^{10(\ln 2)} = 1875 e^{\ln(2^{10})}$$
$$= 1875 (2^{10})$$
$$= 1920000$$

Example 4 You deposit \$500 in a savings account in which interest compounds continuously. After 15 years you have \$850 in the account.

(a) What is the annual rate of interest?

(b) How long does it take for your money to double?

Solution The equation for continuous compound interest is

$$A(t) = C e^{rt}$$ where $C = A(0)$

is the inital amount of money.

Example 4 ... continued

So for this problem we have

$$A(t) = 500e^{rt}$$

and need to find r. We can do this using

$A(15) = 850$.

$$A(15) = 500e^{15r} = 850$$

$$e^{15r} = \frac{850}{500}$$

$$15r = \ln\left(\frac{85}{50}\right)$$

$$r = \frac{1}{15}\ln\left(\frac{17}{10}\right) \doteq 0.035$$

or a 3.5% interest rate.

(b) $$A(t) = 500e^{0.035t} = 1000$$

$$e^{0.035t} = \frac{1000}{500}$$

$$0.035t = \ln(2)$$

$$t = \frac{\ln(2)}{0.035} = 19.80$$

It takes 19.80 years for the money to

7 double.