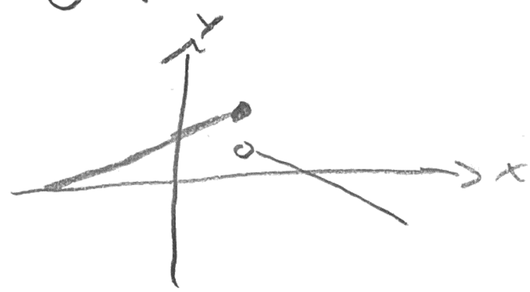
01.19.22

Lesson 4

Finding Limits Analytically

Last time we found limits from the left and right of piecewise defined functions by using a graph.

i.e.



Today we will find limits without a table or a graph.

Continuity $\lim_{x \to c} f(x) = f(c)$ whenever $f(x)$ is continuous at c. For now, that means no jumps, breaks, holes, or asymptotes.

(More on Friday)

Example 1 Let $f(x) = \dfrac{2}{3x-7}$, find $\displaystyle\lim_{x \to \frac{7}{3}} f(x)$

Solution We can do this without a table or graph!

$f\left(\frac{7}{3}\right) = \frac{2}{0}$ so f has a vertical asymptote at $x = \frac{7}{3}$. So the limit as $x \to \frac{7}{3}$ is $+\infty$, $-\infty$, or DNE. Instead of a table we can just check one value smaller than $\frac{7}{3}$ and one larger. The closer to $\frac{7}{3}$ the better.

use $x = \frac{6.999}{3}$ and $x = \frac{7.001}{3}$.

- $f\left(\frac{6.999}{3}\right) = \dfrac{2}{6.999-7} = \dfrac{2}{-.001} = \dfrac{2}{(-)\text{ tiny}}$ so $\displaystyle\lim_{x \to \frac{7}{3}^-} f(x) = -\infty$

- $f\left(\frac{7.001}{3}\right) = \dfrac{2}{7.001-7} = \dfrac{2}{.001} = \dfrac{2}{(+)\text{ tiny}}$ so $\displaystyle\lim_{x \to \frac{7}{3}^+} f(x) = +\infty$

Since $\displaystyle\lim_{x \to \frac{7}{3}^-} f(x) \neq \lim_{x \to \frac{7}{3}^+} f(x)$

$\displaystyle\lim_{x \to \frac{7}{3}} f(x)$ DNE

Example 2 Let $f(x) = \dfrac{x^3 + 11x^2}{x^3 - 17x^2}$, find the following

(a) $\lim\limits_{x \to 0} f(x) = -\dfrac{11}{17}$

$f(x) = \dfrac{x^3+11x^2}{x^3-17x^2} = \dfrac{x^2(x+11)}{x^2(x-17)} = \dfrac{x+11}{x-17}$ $(x \neq 0)$

(b) $\lim\limits_{x \to 17} f(x)$ DNE

$f(0) = \dfrac{0}{0}$ before canceling

(c) $\lim\limits_{x \to -11} f(x) = 0$

(a) $\lim\limits_{x \to 0} f(x) = \dfrac{0+11}{0-17} = \dfrac{-11}{17}$

hole at $x=0$, so after simplifying we can plug in.

(b) $f(17) = \dfrac{17+11}{17-17} = \dfrac{28}{0}$, so vertical asymptote at $x = 17$

check small values

- $f(16.999) = \dfrac{16.999+17}{16.999-17} = \dfrac{\text{number}}{(-)\text{ tiny}}$

so left sided limit is $-\infty$

- $f(17.001) = \dfrac{\text{number}}{17.001-17} = \dfrac{\text{number}}{(+)\text{ tiny}}$

so right sided limit is $+\infty$

Example 2 ...continued

(b) Thus, $\lim_{x \to 17} f(x)$ DNE

(c) Try plugging $x = -11$ in.

$$f(-11) = \frac{-11 + 11}{-11 - 17} = \frac{0}{-28} = 0$$

So $\lim_{x \to -11} f(x) = 0$

Example 3

Let $f(x) = \begin{cases} 2x+1, & x \leq 0 \\ 2x-1, & 0 < x < 2 \\ 4, & x = 2 \\ 3, & x > 2 \end{cases}$

Find the following

(a) $\lim_{x \to 0^-} f(x) = 1$

(b) $\lim_{x \to 0^+} f(x) = -1$

(c) $\lim_{x \to 0} f(x)$ DNE

f(x) is continuous at $x=0$ so to find the left and right limit plug $x=0$ into the correct side

Example 3 ... continued

(d) $\lim_{x \to 2^-} f(x) = 3$

(c) $\lim_{x \to 2^+} f(x) = 3$

(e) $f(3) = 4$

(f) $\lim_{x \to 2} f(x) = 3$

* make sure to plug into the correct part of the piecewise function to see this *

- $x \to 2^-$ goes with $0 < x < 2$
- $x \to 2^+$ goes with $x > 2$

Example 4 Let $f(x) = \begin{cases} x - 20 & x < 25 \\ \sqrt{x} & x \geq 25 \end{cases}$

Find the following limits

(a) $\lim_{x \to 25^-} f(x) = 5$

(b) $\lim_{x \to 25^+} f(x) = 5$

(c) $\lim_{x \to 25} f(x) = 5$

Example 5 Let
$$f(x) = \begin{cases} \frac{x}{\pi} & x \le -\frac{\pi}{2} \\ \frac{1}{2}\sin x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 2x & x \ge \frac{\pi}{2} \end{cases}$$

Find the following Limits

(a) $\lim_{x \to -\frac{\pi}{2}^-} f(x) = -\frac{1}{2}$

(b) $\lim_{x \to -\frac{\pi}{2}^+} f(x) = -\frac{1}{2}$

(c) $\lim_{x \to -\frac{\pi}{2}} f(x) = -\frac{1}{2}$

(e) $\lim_{x \to \frac{\pi}{2}^-} f(x) = \frac{1}{2}$

(f) $\lim_{x \to \frac{\pi}{2}^+} f(x) = \pi$

(g) $\lim_{x \to \frac{\pi}{2}} f(x)$ DNE

- Again, make sure to plug into correct part of the piecewise equation.

$x \to -\frac{\pi}{2}^-$ means $x \le -\frac{\pi}{2}$ etc.