01.24.22

Lesson 6

The Derivative

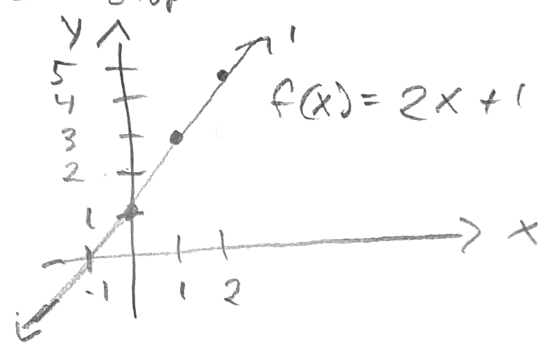
Plane This Week:

Today: Define the derivative as the slope of the tangent line at a point. Use definition

Wed: Basic rules, short cut to using the def.

Fri: Derivative as rate of change

Slope: What is the slope of $f(x) = 2x+1$?



$$\frac{\text{"Rise"}}{\text{Run}} = \frac{f(1) - f(0)}{1-0} = \frac{2(1)+1 - 2(0)+1}{1} = \frac{2}{1} = 2$$

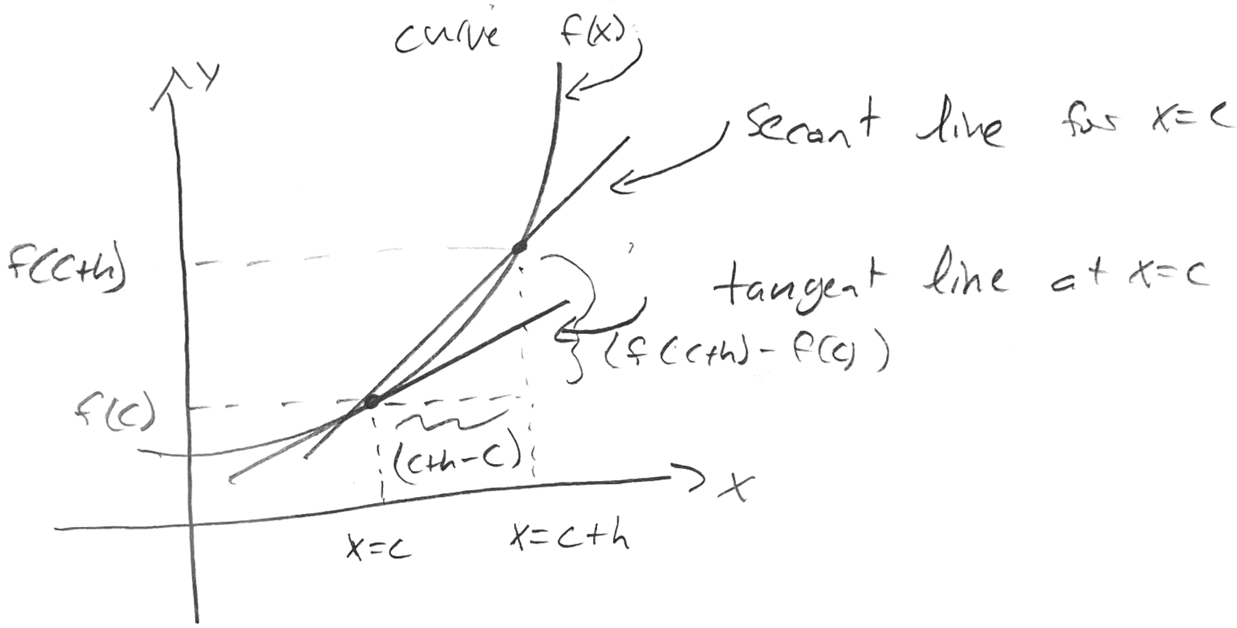
- Can write: slope of a line is $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

- The slope is the same everywhere for a line. But what about a parabola or a curve?

- The slope of a curve at a point $x=c$ is the slope of the tangent line at $x=c$.

- For us, the tangent line at $x=c$ touches the graph at c and nowhere else "close" to c.

- A secant line touches the graph once at c and another point close by.



Slope of secant line: $\dfrac{f(c+h)-f(c)}{c+h-c} = \dfrac{f(c+h)-f(c)}{h}$

As we pick points closer and closer to c, this means we are taking the limit as $h \to 0$

- The Derivative of $f(x)$ at $x=c$ is

slope of Tangent line: $\displaystyle\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

Since this process works for any point $f(x)$ is defined, we get the derivative of $f(x)$ at any point

$$\boxed{\text{Derivative} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

other common notation when $f(x) = y$:

$$f'(x), \; y', \; \frac{dy}{dx}, \; \frac{d}{dx}[f(x)]$$

- In lon-cappa Δx "change in x" instead of h

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- $\frac{f(x+h) - f(x)}{h}$ on its own is called the <u>Difference Quotient</u>

- Find the derivative using the limit process means to use the definition above.

Example 1 Let $f(x) = x^2 + 2$. Find

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 1: Find $f(x+h) = (x+h)^2 + 2$

$$= x^2 + 2xh + h^2 + 2$$

Step 2: Find

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{(x+h)^2 + 2}^{f(x+h)} - \overbrace{(x^2+2)}^{f(x)}}{h}$$

$$= \frac{x^2 + 2xh + h^2 + 2 - x^2 - 2}{h}$$

$$= \frac{2xh + h^2}{h} = 2x + h$$

Step 3: Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h = 2x$

Notes: - Must simplify the top $f(x+h) - f(x)$ before trying to cancle the h on bottom

- we will learn more general methods (short cuts) for finding derivatives, but it's important to understand and practice finding the derivative with a difference quotient.

- Doing so helps us remember where the derivative comes from - A limit!

(Also it will be on the quiz / test)

Example 2 Let $f(x) = \frac{-1}{3x^2}$. Find $f'(x)$ using the limit process.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-1}{3(x+h)^2} - \frac{-1}{3x^2}}{h} = \lim_{h \to 0} \frac{\frac{-1}{3(x+h)^2}\left(\frac{x^2}{x^2}\right) + \frac{1}{3x^2}\left(\frac{(x+h)^2}{(x+h)^2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-x^2 + (x+h)^2}{3(x+h)^2 x^2 h} \qquad \left(\frac{\frac{a}{b}}{h} = \frac{a}{b} \cdot \frac{1}{h} = \frac{a}{bh} \right)$$

$$= \lim_{h \to 0} \frac{-x^2 + x^2 + 2xh + h^2}{3(x+h)^2 x^2 h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{3(x+h)^2 x^2 \cancel{h}} = \lim_{h \to 0} \frac{2x + h}{3(x+h)^2 x^2}$$

← top is fully simplified and the h on the bottom canceled so plug in $h = 0$

$$= \frac{2x}{3(x^2)x^2} = \frac{2}{3x^3}$$

So $\boxed{f'(x) = \frac{2}{3x^3}}$

Example 3 Find the equation of the tangent line to the graph of $f(x) = 5x^2 - 3$ at $x = 2$

Solution step 1) find $f'(2) = m$

step 2) Equation of tangent line

$$(Y - Y_0) = m(X - X_0) \qquad (X_0, Y_0) = (2, f(2))$$
$$(Y - 17) = m(X - 2) \qquad = (2, 17)$$

step 1)
$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h \to 0} \frac{(5(2+h)^2 - 3) - (5(2)^2 - 3)}{h}$$
$$= \lim_{h \to 0} \frac{5(4 + 4h + h^2) \cancel{-3} - 5(4) \cancel{+3}}{h}$$
$$= \lim_{h \to 0} \frac{20h + 5h^2}{h} = \lim_{h \to 0} 20 + 5h = 20$$

So $f'(2) = 20$

Step 2)
$$(Y - 17) = 20(X - 2)$$
$$Y = 20X - 40 + 17$$
$$Y = 20X - 23$$

Example 4 The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)^3 - (x+h)+3} - \sqrt{x^3 - x + 3}}{h}$$

What is $f(x)$?

Answer: $f(x) = \sqrt{x^3 - x + 3}$

How to find out? General Form is always

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So the left piece is $f(x)$

Next class!

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$