LAB 3 (NUMERICAL METHODS)

For this lab you will need the files eul.m, rk2.m and rk4.m as well as the files for dfield and pplane corresponding to the version of MATLAB you are running. These are already present on the Purdue system. If you are using your own copy of MATLAB you will need to down load them. To find them, go to the MA 366 web page and follow the links. This web page also has copies of all of the labs. The address is:

`http://www.math.purdue.edu/~rcp`

Prior to Lab. No specific preparation required.

The purpose of this lab is to explain how computers approximate solutions to differential equations. The first technique to be discussed is Euler's tangent line method. Begin by using dfield to plot the direction field for the equation below

$$y' = y + t$$

for $0 \leq t \leq 3$, $0 \leq y \leq 5$

The idea of the tangent line method is to follow each of the little lines for short distances. Specifically, you could start from an initial position, follow one direction line for a short distance, then pick up another line, follow it a short distance, etc., eventually approximating a solution.

Now, suppose that we want to approximate the solution to

$$y' = t + y \quad y(0) = 0$$

over the interval $0 \leq t \leq 3$.

The TA will demonstrate how to create a function file. For this, you pull down the File menu and select "New M-File". A window will pop-up. In this window, write the following commands, as listed:

```
function yprime=ypt(t,y)
yprime=t+y;
```

We just created a MATLAB function. The TA will demonstrate this by computing ypt(2,3) and ypt(4,5).

Then, (in this window) pull down the "File" menu and select "Close (or Exit)". MATLAB will ask you if you want to save this file. Yes! Save it as "ypt.m".

The TA will then enter the commands

```
hold on
[t,y]=eul('ypt',0,3,0,1);
```

```
plot(t,y,'r');
```

Note: In some versions of dfield, the required syntax is "eul('ypt',0,3,0,1)". Use the command "help eul" to see what the proper syntax is for your version.

You should continue with the following material on your own in the lab before beginning the exercises.

Create the function file for ypt as demonstrated. Then plot the Euler approximation as demonstrated by the TA. What you see is a very crude approximation the a solution to the equation $y' = t + y$ plotted in red. The numbers "[0,3],0,1" (or "0,3,0,1") are, respectively, the initial value of t, the final value of t, the initial value of y and the "step size," which determines how long we follow one tangent line before switching to the next.

You should get a piece-wise linear graph made up of three lines. The first line segment has slope 0 because $y' = t + y$ is zero at $(0, 0)$. Since the step size is 1, we follow this line for 1 unit, arriving at the point $(y, t) = (1, 0)$. The next line has slope 1 since $y' = t+y$ equals 1 at $(1, 0)$. Why is the slope of the third line segment 3?

Next, we study the "improved Euler" method. Look again at the proximate solution you plotted before. (The three line segments.) Note that between $(0, 0)$ and $(1, 0)$, y' changes from 0 to 1. Thus, the average y' over this interval is .5. It turns out that it is more accurate to to use this average value as the direction from $(0, 0)$ rather than the value at $(0, 0)$. Thus, the improved Euler tells us to head away from $(0, 0)$ with slope .5. Enter

```
[t,y]=rk2('ypt',[0,3],0,1);
plot(t,y,'b');
```

Note: In some versions of dfield, the required syntax is "rk2('ypt',0,3,0,1)". Use the command "help eul" to see what the proper syntax is for your version.

"rk2" is a program which computes the Improved Euler approximation. Note that the first line segment does indeed have slope .5. Note also that this line segment ends at $(1, .5)$. What slope the Euler method would suggest at this point? (Ans. 1.5) Where are we if we travel from $(1, .5)$ for one unit at this slope. (Ans. $(2, 2)$.) The Euler slope here is 4. The average Euler slope over this line is then $5.5/2 = 2.75$. This is what the improved Euler method uses as the slope. Note that the second line does indeed rise 2.75 units, eventually ending at $(2, 3.25)$. In general the improved Euler method takes the average of the two Euler slopes and uses that as the improved slope.

There is an even better method (the Runge Kutta method) which is similar to the improved Euler method only even more complicated. It is denoted by rk4 and is used in exactly the same way.

Now you can do the exercises.

Exercises

(1) Exit dfield. Then re-enter dfield and plot the direction field for $y' = y + t$. Use the range $0 \le t \le 3$, $0 \le y \le 8$. Then return to the command window,

execute the "hold on" command, and graph the Euler approximation to $y' = t + y$, $y(0) = 0$ over the interval $0 \leq t \leq 3$ with step size .5. (This graph should appear in the dfield Figure Window.) Use the differential equation to compute the slope of the first three line segments. Show your calculations and get the graph printed.

(2) Graph the Improved Euler approximation to $y' = t + y$, $y(0) = 0$ over the interval $0 \leq t \leq 3$ with step size .5 on the same graph. Use the differential equation to compute the slope of the first three line segments. Show your calculations and get the graph printed.

(3) Use dfield to plot the solution corresponding to $y(0) = 0$. From your graph, what is the value of the maximum error in the Euler and Improved Euler methods (as compared with the plot generated by dfield) for this problem?

(4) Consider the differential equation $y' = (3-y)(y+1)$ with the initial condition $y(0) = 0$. Plot the Euler approximation (with step size .5) to the solution on the same graph as the direction field. Also use dfield to graph the solution. (Use the default range for dfield.) Try to explain why the graph has the "zig-zags". Get the graph printed. For your explanation, note the direction of the slope lines above and below the line $y = 3$. What is the significance of this line?

Note: Your function file will contain lines similar to the ones below

```
function yprime=zz(t,y)
yprime=(3-y)*(y+1);
```

(5) Consider the differential equation $y' = (3 - y)^2(y + 1)$ with the initial condition $y(0) = 0$. Plot the Euler approximation to the solution on the same graph as the direction field. Also use dfield to graph the solution. (You may need to change the default range for dfield.) Why does the Euler method produce such a bad approximation. Explain in similar terms to the explanation above. Try decreasing the step size to see if you can make it work better. Get this graph printed.

(6) Repeat the previous exercise using the Improved Euler method. Decrease the step size until you get what seems to be a reasonable approximation.

(7) Repeat Exercise 5 using rk4 instead of rk2. (This is the "Runga-Kutta" approximation.)

(8) Write a discussion of your conclusions. Address (a) the relative merits of the Euler, Improved Euler and Runga-Kutta methods and (b) what you can conclude about how much faith you can put in numerical solutions. Question: Can you be certain that the graphs produced by dfield are correct? (Dfield uses the Runga-Kutta method with a small step size.)