Lab 4

Prior to Lab. Read the material on the back page.

During the lab. In formula (1) in the attached material, we take $a = 1$, $b = .5$, $c = .75$ and $d = .25$. Thus, the system (1) becomes

$$x' = x - .5xy$$
$$y' = -.75y + .25xy$$

1. Find the equilibrium points for this system.
2. In pplane, select "predator prey" from the gallery menu. Then enter the system. Choose your scale so that the origin is at the center of the picture and the other equilibrium point is centered in the first quadrant. Plot several orbits in each quadrant. It appears that all of the trajectories in the first quadrant are closed curves. (You will see why shortly.) Notice that they all contain the equilibrium point. This is due to a general theorem which says the for a "nice" autonomous system, any closed orbit must contain an equilibrium point. What is the significance for the populations of food fish and selachians of the fact that the orbits are closed loops?
3. The pplane plot is a plot of y against x. Explain why, as a function of x,

$$\frac{dy}{dx} = \frac{-.75y + .25xy}{x - .5xy}$$

Use dfield (*not pplane*) to plot some solutions to this equation for various initial data. Use the same ranges on x and y as Exercise 2. You should find that the solution curves here are the same as the orbits in Exercise 2 except that the top half and the bottom half of each closed orbit must be plotted separately. Can you think of a mathematical reason for this. (Think about the fact that we are representing y as a *function* of x.)

4. The equation in the previous exercise is a separable equation. Solve it to show that the general solution is given implicitly by

$$C = \frac{yx^{.75}}{e^{.5y}e^{.25x}}$$

5. Let $C = 1/2$ in the last exercise. This should describe one orbit.

(a) Approximate the value of y which corresponds to $x = 1$. For this note that for $C = 1/2$ the equation from Exercise 4 is equivalent with

$$\frac{e^{.25x}}{2x^{.75}} = \frac{y}{e^{.5y}}$$

If $x = 1$, the value of the left side is .6420. You can find the corresponding values of y by plotting (using "fplot") the function on the right side of this equality and reading off the appropriate values.

(b) Find the values of y corresponding to $x = 2, 3, 4.., 10$. Explain why you typically get 2 such values (if you get any.) You will eventually find that there is no y value for the given x. Approximate the largest value x_o of x for which there is a corresponding y. (For this, a graph of the *left* side of the equation in (a) might be useful.) Explain why there is only one value of y corresponding to x_o.

(c) Approximate the smallest value x_1 of x for which there is a corresponding y. Explain why there is only one value of y corresponding to x_1.

The fact that y exists only for $x_1 \leq x \leq x_o$ says that the orbit in question lies over the interval $[x_1, x_o]$ on the x-axis. The fact that we get two y values for each x strictly between x_1 and x_o says that the orbit has a top and a bottom. The fact that get only one y value corresponding to x_1 and x_o says that the top and botom curves meet at the end. This is almost a proof that the orbit in question is a closed curve. In fact, it is not hard to complete this analysis to show that the orbit really is a closed curve.

6. In this exercise, we continue part 5 to show why all of the orbits are closed. Let

$$F(x) = \frac{x^{.75}}{e^{.25x}} \text{ and } G(y) = \frac{y}{e^{.5y}}$$

Notice that the the orbits all are described by an equation of the form

$$G(y) = \frac{C}{F(x)}$$

Furthermore, if the orbit begins at a point in the first quadrant, then $C > 0$. (Why?) Plot (using "fplot") the graph of $G(y)$ over the range $0 \leq y \leq 6$. Using your graph, explain why (typically) if there is a positive value of y for which the above equation is valid, then there will be two. These values correspond to the top and bottom of the closed orbit. Explain why $\lim_{x\to\infty} F(x) = 0$. Use this, together with the graph of G, to explain why, for x sufficiently large, there are no values of y for which the above equation is true. Similarly, show that for x sufficiently near 0, there are no such y. This shows that the orbit lies over a closed interval on the x-axis. Thus, each orbit has a top and a bottom and lies over a closed interval in the x-axis.

6. Let us return to our system. Plot (using "Pplane") x and y as functions of t for several different initial conditions from the first quadrant. Do they appear to be periodic. If so, does the period appear to depend upon the initial condition?

7. It can be proved that x and y are indeed periodic. (This follows from the existence and uniqueness theorem together with the fact that the orbits are closed. Ask your classroom instructor for details.) Say that T is the period. We can compute the average values of x and y by the formulas:

$$\overline{x} = \frac{1}{T}\int_0^T x(t)dt \qquad \overline{y} = \frac{1}{T}\int_0^T y(t)dt.$$

Remarkably, we can evaluate these integrals exactly with out knowing either x or y. Give reasons for each of the steps below for evaluating the x integral. Then do the y integral for yourself.

$$\begin{aligned}
\frac{y'}{y} &= -.75 + .25x \\
\frac{1}{T}\int_0^T \frac{y'(t)}{y(t)}dt &= \frac{1}{T}\int_0^T (-.75 + .25x(t))dt \\
\frac{1}{T}\left(\ln|y(T)| - \ln|y(0)|\right) &= -.75 + \frac{1}{T}\int_0^T .25x(t)dt \\
0 &= -.75 + .25\overline{x} \\
3 &= \overline{x}
\end{aligned}$$

Notice that the average value of x does not depend upon how big the orbit is. Where, in your phase plane picture, does the point $(\overline{x}, \overline{y})$ appear?

8. Explain how the system

$$\begin{aligned}
x' &= x - .5xy - \alpha x \\
y' &= -.75y + .25xy - \alpha y
\end{aligned}$$

where α is a small constant describes the effect of fishing on the population. Plot the orbits for a $\alpha = .05$. Are there still closed orbits? Compute the average selachian and food fish populations if $\alpha = .05$ and compare with the average computed above. How does this result relate to the question asked by D' Ancona to Volterra on the back?