Lab 7 Linearization

Prior to lab. No specific preparation is required.

During Lab. In the last lab you classified equlibrium points as as sources, sinks, centers and saddles. (An equlibrium point is a sink if all solutions which begin sufficiently close to it converge to it. It is a source if all solutions sufficiently close to it move away from it. It is a center if all solutions which begin sufficiently close to it "loop around" it. i.e. they return to their intial position after a finite amount of time. Finally, the equilibrium point is a saddle if some solutions converge to it and some move away from it.) You also saw that for a linear system, the eigenvalues of the corrosponding matrix determine what kind of equilibrium point the origin is.

In this lab, we investigate how to determine the nature of the equilibrium points of a non-linear system.

Exercises.

1. Consider the linear system

$$x' = -3x + (\sqrt{2})y$$
$$y' = (\sqrt{2})x - 2y$$

(a) Use pplane with the range $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$ to plot several orbits for this system. What kind of equilibrium point does the origin seem to be? Get your plot printed.

(b) In the pplane "Options" menu, choose "Erase all solutions". Then use pplane with the range $-.1 \leq x \leq .1$ and $-.1 \leq y \leq .1$ to plot several orbits for this system. Get this plot printed. Notice that all of the orbits seem to be tangent to one particular line at the origin.

(c) Let A be the 2×2 matrix which describes this system. Use the command "[X,D]=eig(A)" to find the eigenvalues and eigenvectors for A. Use this information to prove what you observed in (a). How does the line noted in (b) relate to the eigenvectors of A?

2. Consider next

$$x' = -3x + \sqrt{2}y - xy$$
$$y' = \sqrt{2}x - 2y + x^2 + y^2$$

Once again $x = 0$, $y = 0$ is an equilibrium solution.

Repeat parts (a) and (b) from Exercise 1 for this system. In part (a) plot a large enough number of orbits so that it is clear that orbits which do not begin near the origin can behave very differently than the orbits from Exercise 1. You should find, however, that the orbits for part (b) look very much like those for the linear system. Specifically, the line to which they seem to converge should be the same for both systems.

To understand why this happens, suppose that x and y are very small. We would suppose that then xy and $x^2 + y^2$ would be extremely small. In this case, our system should behave like the linear system

$$\begin{aligned} x' &= -3x + \sqrt{2}y \\ y' &= \sqrt{2}x - 2y \end{aligned}$$

which is exactly the linear system considered above. We refer to this as the "approximating" linear system. The matrix A is called the "Jacobian' matrix for the system.

3. Each of the following systems has an equilibrium point at the origin.
(a) Determine the approximating linear system. Find the eigenvalues for this linear system and use them to predict the nature of the equilibrium point at the origin.
(b) Plot some orbits for the approximating linear system to show that what you said in (a) is correct. Turn in the plots.
(c) Plot some orbits for the non-linear system to show that your predictions from (a) also apply to the non-linear system. Turn in the plots.

$$\begin{aligned} x' &= -2x - y - x^3 - yx^2 \\ y' &= x - y + yx^2 + y^3 \end{aligned}$$

$$\begin{aligned} x' &= x(1 - x - y) \\ y' &= y(.75 - y - .5x) \end{aligned}$$

4. A system can have an equilibrium solution which is not at the origin. For example, consider the last system above. If x and y are the constant functions $x = y = .5$ then both equations are satisfied. To study these solutions we introduce the new unknown functions $u = x - .5$ and $v = y - .5$ so $x = u + .5$ and $y = v + .5$. Substituting, we see that the above system becomes

$$\begin{aligned} u' &= -(u + .5)(u + v) = -.5u - .5v - u^2 - uv \\ v' &= -(v + .5)(v + .5u) = -.25u - .5v - v^2 - .5uv \end{aligned}$$

The solutions $x = .5$, $y = .5$ correspond to $u = v = 0$.
(a) Use pplane to plot several orbits for both the x-y system and the linear approximation to the u-v system. Show that when a small enough scale is used, the u-v plot around the origin looks identical with the x-y plot around $(.5, .5)$. Turn in your plots.
(b) Use MATLAB to find the eigenvalues and a basis for the eigenvectors of the approximating linear. Then, using the general solution of the system, prove that the origin is a sink.

5. Our examples so far seem to indicate that the solutions to a non-linear system always behave very much like those of the corresponding linear system as long

as we stay near the equilibrium point. This is not always true. For example, consider the system below.

$$x' = -y - (seed)x^3 \qquad (1)$$
$$y' = x$$

where *seed* is your personal seed number.

(a) Plot a few orbits of the corresponding linear approximation. They should appear to be circles around the origin. Prove that they really are circles. This can be done without solving the system as follows. Let x and y be solutions to the linear approximation. Note that

$$(x^2 + y^2)' = 2xx' + 2yy' \qquad (1)$$

and then use the differential equation to show that this quantity is 0. How does it follow that the orbits are circles?

(b) Use pplane to plot some orbits of the system (1) over the intervals $-2 \le x \le 2$ and $-2 \le y \le 2$. The orbits should seem to spiral in toward the origin, but with a blank spot near the origin.

(c) Change the scale on pplane so that you can see what is happening around the origin. Near the origin the orbits should appear to be loops.

Actually, this is not at all what is happening. Use formula (1) to prove that if x and y are solutions to the differential equation (1), then

$$(x^2 + y^2)' = -(seed)x^4$$

How does it follow that the orbits constantly move toward the origin? How can you explain the fact that the computer output seems to suggest that the orbits are loops while the theory proves that they are spirals?

(d) The orbits constantly move toward the origin, but the graphs certainly indicate that they do not approach the origin. Rather, they seem to approach circles which have a finite radius. This, too, is wrong. The proof is based on an important theorem:

Poincaré-Bendixson Theorem. Suppose we are studying a bounded orbit of an autonomous system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g have continuous first partial derivatives on all of $\mathbb{R}^2$. Suppose that that there is a number $\varepsilon > 0$ such that the distance from the orbit to any equilibrium point of the system is at least ε. Then either the orbit itself is a closed loop, or the orbit approaches an orbit which is a closed loop. Thus, in either case, the system has orbits which are closed loops.

Use your work from part (c) to prove that the system in Exercise 5 has no orbits that are closed loops. How does it follow that each orbit in reality, spirals into the origin?

The examples in this lab demonstrate a theorem. Suppose that we are studying a system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are polynomials with $f(0, 0) = g(0, 0) = 0$. Then the corresponding linear system will have the same type of equilibrium point at the origin as the non-linear system as long as the eigenvalues of the linear system are distinct and not purely imaginary.