LAB 12

In Lab 4 we discussed techniques for approximating solutions of differential equations. In this lab be discuss yet another technique for approximating solutions; Picard iteration. The significance of Picard iteration is that it forms the basis for one proof of the Existence and Uniqueness theorem. The real purpose of this lab is to sketch a proof of this fundamental theorem, in the context of a specific example.

Section 1: Integration

MATLAB has several excellent programs for computing integrals; "trapz" "quad" and "quad8". These programs compute *definite* integrals–i.e. they compute $\int_a^b f(x)\,dx$ where a and b are specific numbers. For the sake of this project, we need to be able to solve problems such as: graph y over the interval $1 \leq t \leq 20$ where

$$y(t) = \int_1^t \cos s^2\,ds \tag{1}$$

However, note that y satisfies the following initial value problem.

$$\frac{dy}{dt} = \cos t^2 \qquad y(1) = 0$$

We can use "ode23" (or "ode45") to find y. Specifically, we create a function file called, say, "coss.m", which describes the function $f(t) = cos(t^2)$. We include y as a variable since ode23 expects a function of both t and y.

```
function u=coss(t,y)
u=cos(t^2);
```

We then set

$$[t, y] = ode23('coss', [1, 20], 0); \tag{2}$$

This computes y over the range $1 \leq t \leq 20$. To plot the integral, we simply execute "plot(t,y)".

We will also need to plot integrals of integrals. For example, suppose that y is as defined in formula (1) and we set

$$z(t) = \int_1^t y(s)\,ds$$

How do we plot z?

One's first thought is to execute "[t,z]=ode23('y', [1,20],0)". This, however, will not work because "y" is a vector, not a function. In fact, $[t, y]$ is a large matrix whose first column is a sequence of t values and whose second column is the corresponding y values. However, the following function file (which we call "yy.m") creates the required function.

```
function u=yy (s,x)
global  t y
u=interp1(t,y,s);
```

The "interp1" command tells MATLAB that if s lies between two entries in the t column, then $yy(s)$ is computed by linearly interpolating between the corresponding entries in the y column.

The "global" command tells MATLAB that for the sake of computing yy, t and y have whatever values you have given them in the Command Window. *Before using yy, you must first restart MATLAB and execute the command "global t y" in the Command Window, which informs MATLAB that you want to make these variables available to functions.*

Finally, to compute the integral of y, you need only execute the command "[t,y]=ode23('yy',[1,20],0);" Note that this will erase the previous values of t and y. If we need to integrate this result, all we need do is reenter this command.

Exercises for Section 1

(1) Use the techniques described above to plot the function $f(t)$ in part (a) below over the range $0 \le t \le 20$. Then, evaluate the integral (by hand) and use "fplot" to plot the answer on the same graph so that you can tell how well MATLAB is doing. Repeat for part (b).
  a) $f(t) = \int_0^t \sin s\, ds$
  b) $f(t) = \int_0^t s^2 e^s\, ds$

(2) For part (b) above, plot the function $g(t) = \int_0^t f(s)\, ds$ using the techniques described in Section 1 above.

(3) In MATLAB, enter "format long" and then compute cos($seed$). Note that MATLAB calls the answer "ans". Next compute cos(ans). Next compute the cosine of this answer. (You need only press the up arrow, followed by "enter".) Keep repeating this process, each time taking the cosine of the previous answer, until the answer seems to stop changing. Notice that you have succeeded in finding a number x_o that satisfies $\cos x_o = x_o$, at least to the accuracy of "format long".

Section 2: Picard Iteration

The process demonstrated in Exercise 3 of Section 1 is called *iteration*. It is useful when we are given a continuous function f and we wish to find an x_o such that $f(x_o) = x_o$. (Such a number is called a "fixed value" because applying f to it does not change it.) We start with a some value x_1 and set $x_2 = f(x_1)$, $x_3 = f(x_2)$ and, in general $x_n = f(x_{n-1})$. If the x_i approach a limit x_o, then x_o will satisfy $f(x_o) = x_o$.

To relate this to differential equations, consider the following initial value problem:

$$y' = t^2 + y^2 \qquad y(0) = y_o \tag{3}$$

Integrating both sides of this equation, and noting that $y(0) = y_o$, we find that y satisfies the following *integral equation.*

$$\begin{aligned} y(t) - y_o &= \int_0^t (s^2 + y(s)^2)\, ds \\ y(t) &= y_o + \int_0^t (s^2 + y(s)^2)\, ds \end{aligned} \tag{6}$$

Conversely, if y is a continuous function that satisfies equation (6), then $y(0) = y_o$ and differentiation of equation (6) results in equation (3). Thus, equation (6) is equivalent with equation (3) in the sense that both equations have the same solutions.

For any continuous function y, let $F(y)$ be the function obtained by substituting y into the right side of the last equation in (6). Equation (6) is equivalent with the equality $y = F(y)$. This is a fixed point problem, involving functions rather than numbers. However, we will approximate our solution in a very similar manner to what we did for the cosine function. Specifically, we start with the constant function $y_1(t) = y_o$. (Note that y_1 satisfies the initial condition in equation (3).) We compute $y_2 = F(y_1)$. Specifically

$$y_2(t) = y_o + \int_0^t (s^2 + y_o^2)\, ds \tag{4}$$

This is the "second Picard iterate." We then let $y_3 = F(y_2)$:

$$y_3(t) = y_o + \int_0^t (s^2 + y_2(s)^2)\, ds \tag{5}$$

producing the third Picard iterate. In general, we define $y_n = F(y_{n-1})$. Our hope is that as n gets large, the y_n become better and better approximations to the actual solution.

Exercises for Section 2

The first series of questions refers to equations (3)-(5).

a) Let $y_o = seed/10$. Use dfield to plot the solution of the initial value problem (3) over $0 \le t \le 4$, $0 \le y \le 20$.

(1) Use MATLAB to compute and plot (in some color other than yellow) y_2 over the same t interval. (The plot will appear in the dfield window.) (*Hint:* Use ode23 to compute the integral in equation (4), calling the output "[t,y]". Then enter "y=y+*seed*/10" .)

b) Use MATLAB to compute and plot $F(y_3)$ in yet another color on the graph from part (a). (*Hint:* You need only change the line "u=interp1(t,y,s)" in the definition of the function yy to "u=s^2+interp1(t,y,s)^2" and then apply ode23 to this function. Don't forget to add y_o onto the integral.)

c) Compute and plot y_4, y_5 and y_6 on this same interval. Do they seem to be approximating the solution? Get this plot printed. Label the graphs so as to indicate which Picard iterates are being plotted.

(2) Let $y_o = seed/10$. Transform the following initial value problem into an integral equation (integrate from 1 to t) and on a single graph plot both the solution and the first 6 Picard iterates over the range $1 \leq t \leq 10$.

$$y' = \frac{y}{t^2 + y^2} \qquad y(1) = seed/10$$

(3) (This is not a computer exercise.) Picard iterates are also useful in proving existence and uniqueness for systems. Let A be an nxn matrix and consider the initial value problem $X' = AX$, $X(0) = X_o$. Transform this equation into an integral equation and show that the second Picard iterate is $X_2(t) = tAX_o + X_o$. What are the third and fourth Picard iterates? What is the n th?

(4) As commented above, our purpose in introducing the Picard iterates is to discuss the proof of the existence and uniqueness theorem. To this end, we will consider the initial value problem

$$y' = t^2 + y^2 \qquad y(0) = 0 \tag{6}$$

The first, and most essential part of the proof is to show that the Picard iterates have a limit y. Once this is known, then then we can proceed to prove that y is continuous and satisfies equation (6), which proves that y is indeed a solution to our initial value problem. In particular, the solution exists. The following discussion proves that the limit exists. The rest of the proof is too long to be considered here.

a) Graph the first three Picard iterates for the initial value problem (6) over the interval $0 \leq t \leq .5$. DO NOT graph the solution to the initial value problem as we are pretending that we do not know it exists. Get your plot printed.

b) On the graph from the previous exercise sketch in (by hand or by computer) the line $y = t/2$. Notice that in the interval $0 \leq t \leq .5$, the Picard iterates all lie below this line and above the t-axis. Prove that this is true for all y_i.

(*Hint:* Since $y_1(t) = 0$ for all t, $0 \leq y_1(t) \leq .5$. It follows that for $0 \leq s \leq .5$, $0 \leq s^2 + y_1(s)^2 \leq .5^2 + .5^2 = .5$. Use this and equation (6) to prove that $y_2(t)$ lies between the t axis and the line. Once you know the result for y_2, you can repeat this argument for y_3, etc.

(5) Prove that for $0 \leq t \leq .5$, $y_{n+1}(t) > y_n(t)$. *Hint:* Use formula (6) to show that

$$y_{n+1}(t) - y_n(t) = \int_0^t y_n(s)^2 - y_{n-1}(s)^2 \, ds$$

(6) It follows that the sequence of values $y_n(t)$ is increasing. An increasing sequence must either converge or tend to ∞. How does the convergence follow?