

## CHAPTER 1

# Compound Interest

### 1. Compound Interest

The simplest example of interest is a loan agreement two children might make: “I will lend you a dollar, but every day you keep it, you owe me one more penny.” In this example, the interest rate is 1%/day and the amount owed after  $t$  days is

$$A(t) = 1 + .01t$$

In this formula, the quantity  $.01t$  is the interest at time  $t$ . (In general, the interest is the difference between what was borrowed and what is owed.)

REMARK. In the above example, we can describe the interest rate as a percent (1%) or as a numeric value (.01). *When we state an interest rate we will always mean a numeric value, and not a percent, unless we indicate otherwise.*

In these notes, we use the year and the dollar as our fundamental units as this is most common in actuarial science. We will assume that, unless otherwise stated, all interest rates are per unit time—i.e. per year. However, the reader should be aware that all of our formulas are valid regardless, of the units of measure.

If, as above, the interest is proportional to time, then we say that the interest is *simple interest*. Thus, if we borrow  $P$  at rate  $i$  simple interest, the amount owed at time  $t$  is

$$A(t) = P + itP = (1 + it)P$$

EXAMPLE 1. On Jan. 1 of a non-leap year, I borrow \$5,000 at 3% simple interest. How much do I owe on May 1? How much would I owe in 3 years?

SOLUTION. On May 1, I have had the money for  $31 + 28 + 31 + 30 = 120$  days, which is 120/365th of a year. Hence, I owe

$$\left(1 + \frac{120}{365}(.03)\right)5000 = 5049.32$$

dollars.

In 3 years, I owe

$$(1 + 3(.03))5000 = 5450.00$$

REMARK. In computing interest, it is typically assumed that interest is earned only on either the first day the account is open or the last day, but not on both. Which day doesn't matter in computing the interest. Thus, in Example 1, it is correct not to count the interest earned on May 1.

The question of how many days are in a year is actually somewhat complicated. The most obvious answer is that a year will have either 365 or 366 days, depending on whether or not it is a leap year. It has to be remembered, however, that accounting practices became standardized long before even hand held calculators

were available, not to mention personal computers. Thus, many schemes have been developed to simplify hand computations.

For example, it is common to not give interest on Feb. 29, in which case all years effectively have 365 days. Another method, referred to as *exact interest*, is to give interest on leap day, but still say that all years have 365 days. Thus, under this standard, at the  $n$ th day of the year,  $P$  dollars grows to

$$\left(1 + \frac{n}{365}i\right)P$$

In particular, at the end of a leap year, you have

$$\left(1 + \frac{366}{365}i\right)P$$

dollars.

There is another method, *ordinary interest*, (not to be confused with “simple interest”) in which it is assumed that all months have 30 days and every year has 360 days! Thus, if you opened an 4% account on Jan. 1 1950 and closed it on May 10, 2002, you held your money for 52 years, 4 months and 10 days which, according to the rules of ordinary interest, is

$$52 \cdot 360 + 4 \cdot 30 + 10 = 18,850$$

days. Hence  $P$  dollars will have grown to

$$\left(1 + \frac{18850}{360}.04\right)P$$

dollars. Ordinary interest has the feature that each month is  $1/12$  of a year.

There is also something called *Banker's rule*, in which every year has 360 days, but you count the exact number of days you have held the money in computing the interest. To use Banker's Rule on the preceding example, you would have to count the days between Jan. 1, 1950 and May 10, 2002 and use this number instead of the 18,850. Good luck!

The use of exact interest is common in Canada while the Banker's rule is common in the US and in international markets. In this class we will always assume that no interest is given on Feb. 29, in which case all years effectively have 365 days. When necessary, we will count the exact number of days, except Feb. 29.

Compound interest is much more common than simple interest. Suppose, for example, that I borrow  $P$  dollars at rate  $i$ , compounded yearly. As with simple interest, at the end of the year, I owe

$$A(1) = (1 + i)P$$

dollars.

With compound interest, however, I pay interest on the total amount owed at the beginning of the compounding period, not just the original principal. Hence, in another year, my debt will again grow by a factor of  $(1 + i)$ , yielding

$$(1 + i)^2 P$$

dollars. After  $n$  years, I owe

$$(1) \quad A(n) = (1 + i)^n P$$

dollars.

In interest theory, the difference between borrowing money and saving money is only in the point of view. When I open a bank account, I am in essence loaning the bank money. The interest I earn on the account is the interest the bank pays me on this loan. Thus, the only difference between a bank loan and a bank account is in who is doing the lending and who is doing the borrowing. In particular, we can analyze savings accounts using the same formulas.

EXAMPLE 2. At the end of 1980, I deposited \$1,000 in an account that earns 7.3% interest, compounded yearly. How much did I have at the end of 2000, assuming that no further deposits or withdrawals are made?

SOLUTION. My funds were on account from Dec. 31, 1980 to Dec. 31, 2000: a full 20 years. Hence, I have

$$(1.073)^{20}1000 = 4,092.55$$

dollars.

EXAMPLE 3. On Jan.1, 1998, I open an account with a \$1000 deposit. On Jan.1, 1999, I withdraw \$500 and on Jan.1, 2001 I deposit \$1,500. If the account earns 7.5% interest, compounded yearly, and no further deposits or withdraws are made, what was the balance on Jan.1, 2003?

SOLUTION. There are two ways to solve this problem; easy and easier. First, the easy way:

The balance on Jan. 1, 1999 was one year interest on \$1000, minus \$500:

$$1000(1.075) - 500 = 575$$

The balance on Jan. 1, 2001 was 2 years interest on \$575, plus the \$1,500 deposit:

$$575(1.075)^2 + 1500 = 2164.48$$

My final balance is 2 years interest on \$2164.48:

$$2164.48(1.075)^2 = 2501.34$$

Now for the easier way. Without any further deposits, our \$1000 would have grown to

$$1000(1.075)^5 = 1435.63$$

Withdrawing \$500 caused us to loose both the \$500 as well as its interest for the next 4 years; a net loss of

$$500(1.075)^4 = 667.73$$

Finally, the \$1,500 deposit was on account for 2 years, yielding a total of

$$1500(1.075)^2 = 1733.44$$

Hence, our balance is

$$1435.63 - 667.73 + 1733.44 = 2501.34$$

as before.

In general, we may treat deposits and withdrawals separately.

EXAMPLE 4. Ed borrows \$550 at 4% interest. At the end of year 1, he pays \$100, at the end of year 2 he pays \$300 and at the end of year 3 he borrows an additional \$50 at the same interest rate. He pays off the loan at the end of year 4. What was his final payment?

SOLUTION. We treat each payment and loan separately. The loans, together with interest, total to a debt of

$$(1.04)^4 550 + (1.04) 50 = 695.42$$

Each payment results in an interest savings. Thus, the payments up to the end of year 4 reduce this debt by

$$(1.04)^3 100 + (1.04)^2 300 = 436.97$$

Thus, Ed still owes

$$695.42 - 436.97 = 258.45$$

which is his last payment.

What if, in Example 2, I were to close my account after having left my money on deposit for only 6 months; how much would I get? The answer depends on the rules of the bank. Some accounts charge a substantial penalty for early withdrawal, meaning that you could actually lose money. In some cases, the bank uses simple interest for partial periods, in which case you would get

$$\left(1 + \frac{.073}{2}\right) 1000 = 1,036.50$$

dollars since the money was on deposit for a half year. Finally, we might simply substitute  $n = 1/2$  into formula (1) yielding

$$(1.073)^{1/2} 1000 = 1,035.86$$

In practice, this last method is probably the least common. However, in the mathematical theory of interest, *if we say that an account earns compound interest at a rate  $i$ , we are implicitly stating that we use formula (1) for partial periods as well:*

DEFINITION 1. *An quantity grows at a rate  $i$  compound interest if the amount at time  $t$  is given by*

$$(2) \quad A(t) = (1 + i)^t P$$

for some constant  $P$ .

EXAMPLE 5. Banks A and B both offer savings accounts that pay 5% interest per year. Bank A compounds yearly but uses simple interest for partial periods while bank B uses straight compound interest for all times. Compare the amount that you would have after 3 years and 2 months if you invested \$2,000 in bank A with the same investment in bank B.

SOLUTION. In bank A, at the end of 3 years, you have

$$(1.05)^3 2000 = 2315.25$$

dollars. For the next 2 months you earn simple interest on \$2,315.25 dollars, yielding

$$\left(1 + .05\left(\frac{2}{12}\right)\right) 2315.25 = 2334.54$$

In bank B you have

$$(1.05)^{3\frac{2}{12}} 2000 = 2334.15$$

This example makes an important point: the difference between using simple interest for partial periods versus compound interest is slight. In fact, in Figure 1 we have graphed the amount of money in banks B and A on the same graph for  $0 \leq t \leq 1$ . The graphs are so close that they appear to be one single graph.

FIGURE 1. There are two graphs here!

The observation that for small time intervals, compound and simple interest are roughly the same is equivalent with saying that for small values of  $t$

$$(3) \quad (1+i)^t \approx 1+it$$

EXAMPLE 6. The following chart is a record of the activity in a certain account that earns compound interest at rate  $i$ . The initial balance was \$50,000 and the final balance was \$48085.44. Approximate  $i$ .

Date	Deposit(+) or Withdraw(-)
Jan. 1	0
May 1	-5000
July. 1	1000
Jan. 1	0

SOLUTION. Since we only need an approximate value of  $i$ , we assume that each month is  $1/12$  of a year. We may treat the effect of each deposit and withdrawal separately. We lost \$5,000, together with its interest for 8 months, and gained \$1,000, together with its interest for 6 months. Thus, using approximation (3),

$$(4) \quad \begin{aligned} 48085.44 &= (1+i)50000 - (1+i)^{8/12}5000 + (1+i)^{6/12}1000 \\ &\approx (1+i)50000 - (1+\frac{2}{3}i)5000 + (1+\frac{i}{2})1000 \\ 48085.44 - 50000 + 5000 - 1000 &\approx (50000 - \frac{2}{3}5000 + \frac{1}{2}1000)i \\ \frac{2085.44}{47166.67} &= .0442 \approx i \end{aligned}$$

REMARK. If we have a calculator (or a computer) with a “solve” command, we can ask it to solve equation (4). Our computer produced  $i = .04419677393$  which agrees favorably with our approximation.

At times, one hears of banks offering accounts which compound at intervals other than one year. For example, a bank might offer an account that pays 6% interest, compounded four times a year. What this means is that every quarter of a year, the account grows by  $\frac{6}{4}\%$ . Thus, in one year,  $P$  dollars grows to

$$\left(1 + \frac{.06}{4}\right)^4 P = (1.0613)P$$

This is the same growth as an account at 6.13% interest, compounded annually. This 6.13% is called the *annual effective yield* while the “6%” interest rate is referred to as the *nominal* rate, in that it’s the rate that the bank might name when describing the account.

In general, the symbol  $i^{(n)}$  indicates a nominal interest rate  $i$  which is compounded  $n$  times a year. Thus, the discussion in the preceding paragraph says that an interest rate of  $.06^{(4)}$  is the same as  $.0613^{(1)}$ . The rate  $i^{(n)}$  is equivalent with the annual effective rate  $j$  where

$$\left(1 + \frac{i^{(n)}}{n}\right)^n = 1 + j$$

EXAMPLE 7. A bank offers an account that yields a nominal rate of return of 3.3% per year, compounded quarterly. What is the annual effective rate of return? How many years will it take for the balance to double?

SOLUTION. Since each year has 4 quarters,  $P$  dollars at the beginning of the year grows to

$$\left(1 + \frac{.033}{4}\right)^4 P = (1.0334)P$$

by the end of the year. Hence, the annual effective rate of interest is 3.34%.

To compute how long it takes for the account to double, we can either work in quarters or years. In quarters, we seek  $n$  so that

$$\begin{aligned} \left(1 + \frac{.033}{4}\right)^n P &= 2P \\ \left(1 + \frac{.033}{4}\right)^n &= 2 \\ n \ln(1.00825) &= \ln 2 \\ n &= \frac{\ln 2}{\ln(1.00825)} = 84.36 \end{aligned}$$

The number of years is  $84.36/4 = 21.09$ .

Since our effective rate of return is 3.34% per year, we can find the answer directly in years as follows:

$$\begin{aligned} (1.0334)^n P &= 2P \\ (1.0334)^n &= 2 \\ n \ln(1.0334) &= \ln 2 \\ n &= \frac{\ln 2}{\ln(1.0334)} = 21.1 \end{aligned}$$

The answer differs slightly from that found previously due to round off error. Specifically, 3.34% is only an approximation of the annual effective yield. A more exact value is 3.3410626%, which does yield the same answer as before.

Actually, both answers might be wrong. If the bank only credits interest each quarter, then the doubling would not occur until after the 85th quarter, in which case the correct answer is  $21\frac{1}{4}$  years.

EXAMPLE 8. Bank A offers a nominal rate of 5.2% interest, compounded twice a year. Bank B offers 5.1% interest, compounded daily. Which is the better deal?

SOLUTION. We convert each nominal rate into an annual effective rate:

**Bank A**

$$\left(1 + \frac{.052}{2}\right)^2 = 1.052676$$

for a 5.2676% annual effective rate of return.

**Bank B**

$$\left(1 + \frac{.051}{365}\right)^{365} = 1.052319134$$

for a 5.2319134% annual effective rate of return. It's darn close, but Bank A wins.

REMARK. Daily compounding is very common. Daily compounding eliminates the problem of partial periods: you get whatever the balance was at the end of the preceding day.

EXAMPLE 9. How much must I deposit today into an account that pays 6.4% to be able to pay you \$500, two years hence?

SOLUTION. Let the amount deposited be  $P$ . We need to solve the equation

$$\begin{aligned}(1.064)^2 P &= 500 \\ P &= (1.064)^{-2} 500 = 441.66\end{aligned}$$

dollars.

The preceding example makes an extremely important point: a promise to pay \$500, two years from today is not worth \$500 today: if we can invest money at 6.4%, \$500 two years from now is only worth \$441.66 today. We say that the *present value* of \$500 two years from now at 6.4% interest is \$441.66. Equivalently, at 6.4% interest, \$441.66 will grow to \$500. Hence, the *future value* of \$441.66 two years from now at 6.4% interest is \$500.

DEFINITION 2. The *future value* ( $FV$ ) of  $P$  dollars at interest rate  $i$ ,  $n$  years from now, is the amount that  $P$  dollars will grow to in  $n$  years. Hence

$$(5) \quad FV = (1 + i)^n P$$

The *present value* ( $PV$ ) of  $P$  dollars at interest rate  $i$ ,  $n$  years from now, is the amount we would need to invest now to yield  $P$  dollars  $n$  years from now. Hence

$$(6) \quad PV = (1 + i)^{-n} P$$

The quantity  $(1 + i)^{-1}$  occurs so often that it has a special symbol:

$$(1 + i)^{-1} = \nu$$

Hence, Formula 6 is often written

$$PV = \nu^n P$$

EXAMPLE 10. On Jan. 1, you won a “\$400,000 sweepstakes.” The prize is to be paid out in 4 yearly installments of \$100,000 each with the first paid immediately. Assuming that you can invest funds at 5% interest compounded yearly, what is the present value of the prize?

SOLUTION. If you invest each \$100,000 payment at 5% interest, in 3 years you will have

$$(1.05)^3 100000 + (1.05)^2 100000 + (1.05) 100000 + 100000 = 431012.50$$

The present value is

$$(1.05)^{-3} 431012.50 = 372324.80$$

dollars. Note that the award is actually worth considerably less than the advertised \$400,000.

REMARK. We could have done this calculation in one step:

$$\begin{aligned} & (1.05)^{-3} ((1.05)^3 100000 + (1.05)^2 100000 + (1.05) 100000 + 100000) \\ &= 100000 + (1.05)^{-1} 100000 + (1.05)^{-2} 100000 + (1.05)^{-3} 100000 = 372324.80 \\ &= 100000 + \nu 100000 + \nu^2 100000 + \nu^3 100000 = 372324.80 \end{aligned}$$

In the last equality we are just summing the present value of each payment.

The single most important theorem in interest theory is

$$(7) \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

The proof is simple:

$$\begin{aligned} (1 + x + x^2 + \cdots + x^n)(1 - x) &= (1 + x + x^2 + \cdots + x^n)1 \\ &\quad - (1 + x + x^2 + \cdots + x^{n-1} + x^n)x \\ &= 1 + x + x^2 + \cdots + x^n \\ &\quad - x - x^2 - \cdots - x^n - x^{n+1} = 1 - x^{n+1} \end{aligned}$$

which is equivalent with formula (7).

EXAMPLE 11. I deposited \$300 at the end of each year from 1981 to 2000 into an account that yields 3% interest per year. How much do I have at the end of 2000?

SOLUTION. I made a total of 20 deposits. My first deposit earned interest for 19 years and my last deposit earned no interest at all. Hence, I received

$$300(1.03)^{19} + 300(1.03)^{18} + \cdots + 300 = 300((1.03)^{19} + (1.03)^{18} + \cdots + 1)$$

dollars.

From formula (7), with  $x = 1.03$ , this equals

$$300 \frac{(1.03)^{20} - 1}{.03} = 8061.11$$

dollars.

EXAMPLE 12. I deposited \$300 at the *beginning* of each year from 1981 to 2000 into an account that yields 3% interest per year. How much do I have at the end of 2000?

SOLUTION. Depositing at the beginning of a given year is the same as depositing at the end of the preceding year. Thus, we may consider that I deposited \$300 at the end of each year from 1980 to 1999 for a total of 20 deposits. From the reasoning of Example 11, at the end of 1999, we had \$8061.11. At the end of 2000 we gain one year's interest on this amount—i.e. we have

$$(1.03)8061.11 = 8302.95$$

dollars.

An account into which we make either periodic deposits (as in Examples 11 and 12) or periodic withdrawals is called an *annuity*. If the transactions always occur at the end of the compounding period, as in Example 11, the annuity is said to be an *annuity immediate* while if the transactions always occur at the beginning of the compounding period, as in Example 12, the annuity is said to be an *annuity due*.

If we deposit  $D$  dollars per year at the end of the year for each of  $n$  years, then, from formula 7, we have

$$A(n) = (1+i)^{n-1}D + (1+i)^{n-2}D + \cdots + (1+i)D + D = \frac{(1+i)^n - 1}{i}D$$

dollars at the end of the  $n$ th year, which we may express as

$$A(n) = s_{\overline{n}|i}D$$

where

$$(8) \quad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

If the deposits are made at the beginning of the year, then the formula is the same, except that we have an additional year's interest:

$$A(n) = (1+i) \frac{(1+i)^n - 1}{i}D$$

dollars at the end of the  $n$ th year, which we may express as

$$A(n) = (1+i)s_{\overline{n}|i}D$$

We may use formula (8) for accounts with non-yearly compounding. We only need to remember that  $i$  is the interest per period and  $n$  is the number of periods.

EXAMPLE 13. At the beginning of 1992, I opened a bank account earning 4% compound interest with a \$5,000 deposit. I deposited \$100 at the end of each month from 1992 to 2001. What was my account worth at the end of 2001, after my last deposit?

SOLUTION. Over the 10 years from Jan. 1, 1992 to Dec. 31, 2001, the original \$5000 grew to

$$(1.04)^{10}5000 = 7401.22$$

Each month, my account grows by

$$(1.04)^{1/12} = 1.003273740$$

making the monthly rate .003273740. Ten years is the same as 120 months. Hence, from formula (8), the monthly deposits accumulated to

$$\frac{(1.003273740)^{120} - 1}{.003273740}100 = 14669.59$$

Hence, the total is

$$7401.22 + 14669.59 = 22070.81$$

dollars.

REMARK. Computations using short compounding periods over long periods of time are highly susceptible to round-off error. For example, had we rounded the monthly rate from Example (13) to .003, then we would compute the accumulation of the monthly deposits as

$$\frac{(1.003)^{120} - 1}{.003} 100 = 14418.57$$

which is off by more than \$250! When doing compound interest problems, you should make full use of the memory of your calculator, writing as little on paper as possible.

In general, if an account earning interest at rate  $i$  initially has  $P$  and we deposit  $D$  per year, at the end of  $n$  years, we have

$$A(n) = (1 + i)^n P + s_{\overline{n}|i} D$$

In actuarial work, the most typical example of an annuity is a retirement account where the individual accumulates a sum of money while employed, intending to make periodic withdrawals over a space of time to cover living expenses. Mathematically, a withdrawal is just a negative deposit. Hence, if an account earning interest at rate  $i$  initially has  $P$  and we withdraw  $D$  per year, at the end of  $n$  years, we have

$$(9) \quad A(n) = (1 + i)^n P - s_{\overline{n}|i} D$$

EXAMPLE 14. I plan to retire at age 70, at which time I will withdraw \$5,000 per month for 20 years from my IRA. Assuming that my funds are invested at 4.7% interest, how much must I have accumulated in my IRA?

SOLUTION. Let the amount in the account be  $P$ . Since we are making monthly withdrawals, we first must compute the monthly interest rate:

$$j = (1.047)^{1/12} - 1$$

which we store in the memory of our calculator. From equation (9), for my account to be totally depleted in 240 months years,

$$\begin{aligned} 0 &= (1 + j)^{240} P - \frac{(1 + j)^{240} - 1}{j} 5000 \\ &= (2.5057263)P - 1,963,267.91 \\ P &= \frac{1,963,267.91}{2.5057263} = 783,512.51 \end{aligned}$$

dollars

We commented that depositing money in a bank is the same as lending the money to the bank. Withdrawing money from the bank is the same as the bank repaying part of the loan. It follows that formula (9) also describes the amount you owe on a loan after  $n$  payments where  $P$  is the amount borrowed (the principal),  $D$  is the amount of each payment, and  $i$  is the interest. Or, to explain this in different terms, if you borrow  $P$  dollars at rate  $i$ , then after the first payment, you owe

$$(1 + i)P - D$$

After the second payment you owe

$$(1+i)((1+i)P - D) - D = (1+i)^2P - ((1+i) + 1)D$$

By the same reasoning, after the  $n$ th payment you owe

$$A(n) = (1+i)^n P - ((1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1)D = (1+i)^n P - a_{\overline{n}|i}D$$

which is exactly formula (9).

EXAMPLE 15. I borrow \$25,000 to buy a car on which I pay \$1000 down and make monthly payments at the end of the month over the next 5 years. If I pay 7% interest, compounded monthly, what are my monthly payments?

SOLUTION. After my down payment, I owe \$24,000. Saying that the interest is 7% compounded monthly, means that the monthly interest rate is  $.07/12$ . From formula (9)

$$\begin{aligned} 0 &= \left(1 + \frac{.07}{12}\right)^{5 \cdot 12} 24000 - \frac{\left(1 + \frac{.07}{12}\right)^{12 \cdot 5} - 1}{.07/12} D \\ 0 &= 34023.01 - 71.59D \\ D &= 475.23 \end{aligned}$$

which is our monthly payment.

EXAMPLE 16. I make the following deal with a piano rental company. For \$200 a month, I can rent a piano which is worth \$15,000. After 10 years, I own the piano. In essence, they are loaning me \$15,000 which I repay in installments of \$200/month. What annual interest rate are they charging me for this loan?

SOLUTION. Let  $i$  be the monthly interest rate. From formula (9), for my loan to be paid off in 10 years

$$(10) \quad 0 = (1+i)^{120} 15000 - (((1+i)^{120} - 1)/i) 200$$

There is no direct way to solve this equation for  $i$ . Most financial calculators have programs that will solve such equations. Lacking a financial calculator, the best approach is “trial and error.” To set this up on the TI 30XII calculator, I would first guess an interest rate. My favorite choice is  $.06$ , which is a  $.005$  monthly rate. I would store this in the “Ans” register by entering “.005=” into my calculator. Next, I would compute the expression on the right of Formula 10, using the [2nd][Ans] key for each occurrence of “ $i$ ”. In this case, I get  $-5484.92$ . The observation that this is a large negative number tells me that at  $i = .005$  per month, my \$ 200 payments accumulate to much more than \$ 15,000. Hence, my assumed rate is much too low.

I next try  $i = .006$ . For this, I enter “.006=” into my calculator to change the value in the “Ans” register. Then I simply press the up arrow button to return to the formula, and press “=” to compute the expression on the right of Formula 10 for this new interest rate, getting  $-4250.33$ ; so  $.006$  is too low. It turns out that  $.007$  and  $.008$  are still too low and  $.009$  is too big.  $.0085$  yields  $-22.57$ , which is close to 0.  $.00851$  is too small and  $.00852$  is too big. Hence, to three digits, our answer is  $.00851$  which is equivalent to 10.2% converted monthly.

EXAMPLE 17. For a price, I will sign a contract to pay you \$250 at the end of each quarter next 9 years, together with an additional \$10,000 payment at the end of the 9th year. What is the most you should pay for this contract, granted that you can invest funds at 7.1% interest.

SOLUTION. Each quarter, our account grows by a factor of

$$(1.071)^{1/4} = 1.017296072$$

making the quarterly rate

$$.017296072$$

If you invest my payments, at the end of the 9 years you will get

$$\frac{(1.017296072)^{9 \cdot 4} - 1}{.017296072} 250 + 10000 = 22343.57$$

The present value is

$$(1.017296072)^{-36} 22343.57 = 12051.67$$

This is the most you should pay.

REMARK. A coupon bond is an investment that typically yields a fixed sum (referred to as the *coupon*) every quarter for a period of years (the *term* of the bond) together with a lump sum payment (the *redemption value*) at the end of the term. Thus, Example 17 could represent a 9 year bond with a \$10,000 redemption value and \$250 coupons. The \$12,051.67 is the highest price we should be willing to pay for this bond, granted that we can invest at 7.1%.

It is often said that as stock prices drop, bond prices rise. It is easy to understand why. Since we cannot invest money at as high of a rate of return, the idea of a steady income of \$250 together with \$10,000 at the end appears much more attractive and hence is worth more money.

EXAMPLE 18. Suppose that in Example 17, we were only able to invest money at 5% interest. How much should we pay for the same contract?

SOLUTION. Now the quarterly interest rate drops to  $j$  where

$$1 + j = (1.05)^{1/4} = 1.012272234$$

and our investments accumulate to

$$\frac{(1.012272234)^{9 \cdot 4} - 1}{.012272234} 250 + 10000 = 21231.21$$

The present value is

$$(1.012272234)^{-36} 21231.21 = 13685.83$$

Hence, the value of the contract has increased by over \$1,000.

### Exercises

Calculate each of the following:

- (1) You invest \$500 at 6.4% simple interest per year.
  - (a) How much is in your account after 1 year?
  - (b) How much is in your account after 5 years?
  - (c) How much is in your account after 1/2 year?
  - (d) How much is in your account after 5 years and 6 months?
- (2) Redo Exercise 1 using *compound* interest instead of simple interest.
- (3) Redo Exercise 1 assuming that the account earns *compound* interest for integral time periods and simple interest for fractional time periods.

- (4) How long will it take for \$1,000 to accumulate to \$2000 at 5% annual compound interest?
- (5) Value of \$1250 invested for 4 years at 5% simple interest.
- (6) Value of \$1250 invested for 4 years at 5% compound interest.
- (7) Value of \$624 invested for 3 years at 6% simple interest.
- (8) Value of \$624 invested for 3 years at 6% compound interest .
- (9) Value of \$624 invested for 3 years at 6% compounded quarterly.
- (10) Value of \$3150 invested for 1 year at 4% simple interest.
- (11) Value of \$3150 invested for 1 year at 4% compound interest.
- (12) Value of \$8635 invested for 8 years at 5% compounded monthly.
- (13) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% simple interest.
- (14) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compound interest .
- (15) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compounded monthly.
- (16) Amount you need to invest now to have \$100,000 in 15 years if your account pays 5% compounded monthly.
- (17) Your account had \$486 in it on October 1, 1989 and \$743 in it on October 1, 1997. Assuming that no additions or withdrawals were made in the meantime, what annual effective interest rate accounts for the growth in the balance?
- (18) What is the annual effective interest rate that would account for a CD increasing in value from \$4000 on October 1, 1992 to a value of \$5431.66 on October 1, 1997?
- (19) One bank is paying 4.8% compounded monthly. Another bank is paying 5% annual effective. Which is paying more?
- (20) What is the annual effective rate on an investment that is paying 6% compounded quarterly?
- (21) What is the present value of a payment of \$12,000 to be made at the end of 6 years if the interest rate is 7% effective?
- (22) What is the value in eight years (i.e. the future value) of a payment now of \$45,000 if the interest rate is 4.5% effective?
- (23) George is borrowing \$20,000 and will pay 8.5% interest . He will pay off the loan in three annual installments, \$6,000 at the end of the first year, \$7,000 at the end of the second year, and a final payment at the end of the third year. What should the amount of the final payment be?
- (24) Alice is saving money for a new car she hopes to buy in four years. She is putting the money in an account that earns 5.5% effective. At the beginning of this year, she deposited \$4,000, and the beginning of the second year, she expects to deposit \$5,000, and the beginning of the third year, she expects to deposit \$5,000 again. She anticipates the price of the car she wants to buy will be \$20,000. How much more will she need in four years to make the purchase?
- (25) Value of \$12,500 invested for 3 years at 6% compounded quarterly.
- (26) Amount you need to invest now to have \$25,000 in 4 years if your account pays 6% compounded monthly.

- (27) You can buy a \$25 Series EE savings bond for \$18.75; that is, you invest \$18.75 today and in 6 years, you get back \$25. What is the annual effective interest rate that you are getting.
- (28) Amount you will have if you invest \$75 at the end of each month for 10 years if the account pays 7.5% compounded monthly.
- (29) Amount you will have on December 31, 2000 if you invested \$150 at the end of each month starting in January 1996 in an account that pays 4.8% compounded monthly.
- (30) Amount you can borrow today if you are willing to pay \$300 at the end of each month for 5 years for a loan that charges 9% (compounded monthly).
- (31) Amount you can borrow on January 15, 2001 if you are willing to pay \$1250 on the 15th of each month, starting February 15, 2001 with the final payment on July 15, 2013 if the interest rate is 6% compounded monthly.
- (32) Amount you need to invest at the end of each month to have saved \$8000 at the end of 4 years if the account pays 5% compounded monthly.
- (33) Amount of each payment if you borrow \$18000 on a 60 month auto loan that is charging 10.8% (compounded monthly).
- (34) Amount of each monthly payment on a 15 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500.
- (35) Amount of each monthly payment on a 30 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500. (Compare with previous problem.)
- (36) You intend to start depositing \$100 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly?
- (37) You intend to start depositing \$200 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly? (Compare with previous problem.)
- (38) Bob Roarman will sell you a slightly used car for \$7,500 cash or you can buy the same car for 60 payments of only \$159 each (made at the end of each month). What rate of interest is Bob charging?
- (39) Starting January 31, 1990, you saved \$100 at the end of each month in an account that paid 6% compounded monthly. Starting August 31, 1999, you increased the monthly contribution to \$150. How much will you have accumulated by December 31, 2000?
- (40) Assuming that house prices have inflated at an average rate of 8% per year for each of the last 20 years, how much would a house that is currently worth \$100,000 have cost 20 years ago?
- (41) On Jan. 1, 1998, I opened an account in a bank yielding 3.4% annual (compound) interest. On each subsequent Jan. 1, I made either a deposit or a withdraw according to the following chart. What is my balance on Dec. 31, 2002?

Year	1998	1999	2000	2001	2002
De./With.	2000	-700	600	-200	1500

- (42) I borrow \$5,000 at 7.1% compound interest per year for 5 years with yearly payments starting at the end of the first year . My first 4 payments were: \$1,000, \$700, \$2,000, and \$1,000. What is my last payment?
- (43) I want to be able to buy a \$25,000 car in ten years. If I can invest money at 8% annual effective interest, how much do I need to invest now?
- (44) What is the present value of \$25,000 ten years from now at 8% interest?
- (45) If I invest \$25,000 now at 8% annual effective interest, how much will I have in ten years?
- (46) What is the future value of \$25,000 ten years from now at 8% interest?
- (47) I buy a piano from Cheapside Music company on March 1, 1998. I pay \$1,000 immediately, then 3 more payments of \$1,000 on March 1 for each of the next 3 years. Finally, on March 1, 2002, I pay \$10,000. What was this deal worth to Cheapside on March 1, 1998, assuming that they can invest funds at 4% interest—i.e. what was the present value of all of my payments on March 1, 1998?
- (48) In Exercise 47, assuming that Cheapside did invest all of my payments at 4% interest, how much did they have in their account on March 1, 2002?
- (49) You just won the Publisher Clearing House grand prize which is \$1,000,000 paid in 10 annual installments of \$100,000 each. The first installment is paid immediately. Assuming that you can invest money at 3.9% annual effective interest, what is the present value of this prize?
- (50) An account earning annual effective interest  $i$  had a balance of \$ 1500 on Jan. 1, 2000 and a balance of \$1773.63 on Jan. 1, 2001. The year's activity in the account consisted of (a) a deposit of \$ 500 on June 1, a withdrawal of \$ 600 on Aug. 1 and a deposit of \$ 300 on Dec. 1. Approximate  $i$  using the technique from Example 6.
- (51) An insurance company earns 7% on their investments. How much must they have on reserve (the present value of claims) on January 1, 2002 to cover the claims for the next 3 years, if they expect claims of \$500,000 for 2002, \$300,000 for 2003 and \$250,000 for 2004. For sake of simplicity, assume that the claims are all paid on Dec. 31 of the stated year.
- (52) On January 1, 1995, Susan put \$5,000 into a bank account at Stingy's Bank which pays 3.5% compounded twice a year. On July1, 1996, she withdrew \$500. On January 1, 1997 she deposited an additional \$700. How much did she have on account on January 1, 2000?
- (53) An interest rate of 6% compounded three times a year is equivalent to what rate of interest compounded twice a year.
- (54) I have a bank account that initially has \$6,000. After 2 years I withdraw \$4,000. After 2 more years (4 years total), I empty the account by withdrawing \$3,000. What is the annual effective interest rate  $i$ ?
- (55) Re-do Exercise 28 under the assumption that the account earns 7.5% ANNUAL EFFECTIVE. Hint: In this case, each month funds on deposite grow by a factor of  $(1.075)^{(1/12)}$ .
- (56) Re-do exercise 36 under the assumption that the account earns 6% compounded quarterly.
- (57) Re-do 37 under the assumption that the account earns 6% compounded quarterly.

- (58) How much is a 15 year bond that pays quarterly coupons of \$75 and has a face value of \$5,000 worth, assuming that you can invest funds at 3%, compounded quarterly?—i.e. what is the present value at 3% compounded quarterly, of 60 quarterly payments of \$75 plus a final payment of \$5,000?
- (59) How much can you borrow at 3% interest, compounded quarterly, if you can pay \$75 each quarter for 15 years, together with a final "balloon" payment of \$5,000?

## 2. Rate of Return

In finance, one of the most fundamental problems is determining the rate of return on a given investment over a given year, the *annual effective rate of return*. In the simplest case, we put money in at the beginning of the year and don't touch it until the end of the year. In this case, the annual effective rate of return  $i$  is defined by

$$(11) \quad 1 + i = \frac{B_1}{B_0}$$

where  $B_1$  is the end balance and  $B_0$  is the initial balance. Thus, the annual effective rate of return, as a percentage, is the percentage of growth in the account over the year. (Of course, it can also be negative, in which case it represents the percent of decrease in the account.)

The problem of determining the rate of return becomes more complicated when money is being added to or subtracted from the account throughout the year. This situation might arise, for example, in a large mutual fund in which individual investors periodically buy and sell shares. In this case, there are several different ways of defining the rate of return which do not necessarily yield the same answer. One of the most common is the following.

**DEFINITION 3.** The *annual effective rate of return* (or *dollar weighted rate of return*) on an investment over a given year is that interest rate  $i$  which would yield the same final balance for the same activity in the account. It is found by solving the equation

$$B_1 = (1 + i)B_0 + (1 + i)^{1-t_1} D_1 + (1 + i)^{1-t_2} D_2 + \cdots + (1 + i)^{1-t_k} D_k$$

for  $i$ , where  $B_0$  is the initial balance,  $B_1$  is the final balance, the  $D_k$  are the values of the deposits or withdrawals and the  $t_k$  are the times (in fractions of a year) of the deposits, where withdrawals are considered as negative deposits.

Thus, in Example 6 in Section 1, we determined the annual effective rate of return on the account. Here is another example of the same concept.

**EXAMPLE 19.** The following chart is a record of the activity in an investment. The initial balance was \$10,000 and the final balance was \$10,176.22. Determine the annual effective rate of return.

Date	Deposit(+) or Withdraw(-)
Jan. 1	0
Apr. 1	1500
Sept. 1	-1000
Jan. 1	0

SOLUTION. We need to solve the following equation for  $i$ .

$$\begin{aligned}
 (12) \quad 10176.22 &= (1+i)10000 + (1+i)^{9/12}1500 - (1+i)^{4/12}1000 \\
 &\approx (1+i)10000 + (1 + \frac{3}{4}i)1500 - (1 + \frac{i}{3})1000 \\
 10176.22 - 10000 - 1500 + 1000 &\approx (10000 + \frac{3}{4}1500 - \frac{1}{3}1000)i \\
 \frac{-323.78}{47166.67} &= -.030 \approx i
 \end{aligned}$$

Hence, the annual effective rate of return was  $-3\%$ .

The annual effective rate of return is, in a way, the average interest rate on the investment over the year. Thus, in Example 19, if we invested \$1 at the beginning of the year, then we expect that our investment would only be worth around \$.97 at the end of the year. This will be exact if the interest rate stayed constant over the year—i.e. the value at time  $t$  is given by formula (2). In a typical investment, this will only be approximately correct. In fact, in an extremely volatile market where the return rates fluctuate wildly, it might be very far from true.

### 3. Discount and Force of Interest

According to formula 6, the value of money decreases as we look backward in time. Specifically, at rate  $i$ ,  $P$  dollars today was worth only  $(1+i)^{-1}P$  dollars last year. The amount of decrease, then, is

$$P - (1+i)^{-1}P = (1-\nu)P$$

The rate of decrease

$$(13) \quad d = 1 - \nu$$

is called the *discount rate*. It is common to state the discount rate, instead of the interest rate, in which case we will typically need first to compute the interest rate. Formula 13 may be written in the form

$$(14) \quad (1+i)^{-1} = 1 - d$$

which facilitates translating back and forth between discount and interest. Just as with interest, there are also nominal discount rates.

EXAMPLE 20. How much will we have after 10 years if we invest \$ 2000 at a discount rate of 4% per year, compounded monthly.

SOLUTION. From Formula 14, the monthly interest rate is computed from

$$1 + i = (1 - \frac{.04}{12})^{-1} = 1.003344482$$

Hence  $i = .003344482$ . Thus, using Formula 5, our accumulation is

$$(1.003344482)^{120}2000 = 2985.64$$

dollars.

Another common measure of interest is what is called the “force of interest.” From Formula 5, the amount  $A(t)$  in the account at time  $t$  satisfies

$$\begin{aligned} A(t) &= (1+i)^t P \\ (15) \quad &= e^{\ln((1+i)^t)} P \\ &= e^{t\delta} P \end{aligned}$$

where

$$(16) \quad \delta = \ln(1+i)$$

The number  $\delta$  is the *force of interest*.

EXAMPLE 21. An account grows with a force of interest of .0334 per year. What is the interest rate?

SOLUTION. From Formula 16,

$$1+i = e^\delta = e^{.0334} = 1.034998523$$

Thus,  $i = .035$ .

Note that differentiating Formula 15 produces

$$A'(t) = \delta A(t)$$

This tells us two important things:

- (1) The rate of growth of the amount function is proportional to the amount of money in the account.
- (2) The proportionality constant is the force of interest.

We can use the force of interest to describe circumstances in which the interest rate varies over time. Specifically, solving the preceding equation for  $\delta$  produces

$$(17) \quad \delta = \frac{A'(t)}{A(t)} = \frac{d}{dt}(\ln A(t))$$

DEFINITION 4. If  $A(t)$  represents the amount in an account at time  $t$ , then the force of interest for this account is the quantity  $\delta(t)$  defined by Formula 17.

Note that from Formula 17

$$\begin{aligned} \int_{t_0}^{t_1} \delta(t) dt &= \ln A(t_1) - \ln A(t_0) \\ &= \ln \frac{A(t_1)}{A(t_0)} \end{aligned}$$

This formula proves the following proposition.

PROPOSITION 1. *If an account grows with force of interest  $\delta(t)$  over the interval  $[t_0, t_1]$ , then*

$$(18) \quad A(t_1) = e^{\int_{t_0}^{t_1} \delta(t) dt} A(t_0)$$

EXAMPLE 22. What is the annual effective rate of return on an account that grew at 4% interest per year for the first 2 years, a force of interest of  $\delta_t = \frac{1}{2+t}$  for the next 3 years, and a discount rate of 4% for the last 2 years?

SOLUTION. From formulas (18) and (14) over the 7 year period  $P$  dollars will grow to

$$\begin{aligned} & (1 - .04)^{-2} e^{\int_2^5 \frac{1}{2+i} dt} (1.04)^2 P \\ &= (1.085) e^{(\ln 7 - \ln 4)} (1.0816) P \\ &= (1.085) \left(\frac{7}{4}\right) (1.0816) P \\ &= (2.054) P \end{aligned}$$

The annual effective rate  $i$  is determined by solving the equation

$$(1 + i)^7 = 2.054$$

which yields  $i = 10.82\%$ .