100

$3x - 2y + z = 4$

1. Find the normal vector to the plane $3x - 2y = 4 - z$. 4 pts

$\vec{n} = 3\hat{i} - 2\hat{j} + \hat{k}$

2. Find the point of intersection of the lines L_1 and L_2 described the vector equations

$$\mathbf{r}_1(t) = -5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} + t(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$\mathbf{r}_2(t) = 4\mathbf{i} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j})$$

6 pts

$x_1 = -5 + 2t$ $\quad x_2 = 4 + 3s$

$y_1 = -2 + t$ $\quad y_2 = -s$

$-2 + 3 = -s$

$s = -1$

$-5 + 2t = 4 + 3s$

$-2 + t = -s$ $\quad s = 2 - t$

$$\begin{matrix} -5 & 6 & 1 \\ -2 & 3 & 1 \\ -2 & 3 & 1 \end{matrix}$$

$-5 + 2t = 4 + 3(2 - t)$

$-5 + 2t = 4 + 6 - 3t$

$5t = 15$

$t = 3$

$$\begin{matrix} 4 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$$

$x_1(3) = -5 + 2(3) = 1$

$y_1(3) = -2 + 3 = 1$

$z_1(3) = -2 + 3 = 1$

point of intersection: $(1, 1, 1)$

100

3. Find an equation which describes the plane that contains both the point $(2, 1, 1)$ and the line defined by $\mathbf{r}(t) = 4\mathbf{i} + \mathbf{k} + t(3\mathbf{i} - \mathbf{j})$. 10 pts

$x = 4 + 3t$ $\quad t = 0 \quad$ P: $x = 4$, $y = 0$, $z = 1$

$y = -t$

$z = 1$

$t = 1 \quad$ Q: $x = 7$, $y = -1$, $z = 1$

R: $x = 2$, $y = 1$, $z = 1$

$PQ = \langle 7, -1, 1 \rangle - \langle 4, 0, 1 \rangle$

$= \langle 3, -1, 0 \rangle$

$PR = \langle 2, 1, 1 \rangle - \langle 4, 0, 1 \rangle$

$= \langle -2, 1, 0 \rangle$

$$PQ \times PR = \begin{vmatrix} i & j & k \\ 3 & -1 & 0 \\ -2 & 1 & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (3-2)k$$

$= k$

$0x + 0y + z = 1$

plane equation: $z = 1$

100

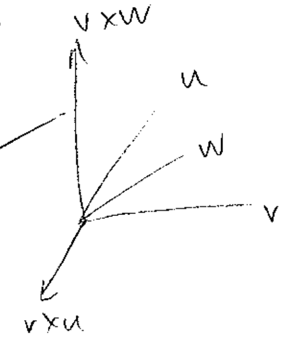
4. Which of the following statements are true for all non-zero vectors **u** and **v** and **w**? In each case justify your answer. 8 pts

   (a) If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

   (b) If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ then $\mathbf{v} = \mathbf{w}$.

   (c) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$.

   (d) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.



a) true; $(u \times v) \cdot w = (u \times w) \cdot w = 0$

The cross product $u \times w$ is orthogonal to u and w. The dot product of two perpendicular nonzero vectors is zero.

b) false; $u \times v = u \times w$ $\quad |u||v| \sin\theta_v \vec{n} = |u||w| \sin\theta_w \vec{n}$

($\vec{n}$ equal for both) $\quad |u||v| \sin\theta_v = |u||w| \sin\theta_w$

$|v| \sin\theta_v = |w| \sin\theta_w \quad |v| = \dfrac{|w| \sin\theta_w}{\sin\theta_v}$, which is not necessarily equal to $|w|$. Hence v is not always = to w.

SEE OTHER ~~SIDE~~ below

c) true. geometrically, $u \cdot (v \times w)$ is the volume of the parallelepiped determined by u, v, and w. Likewise, $(v \times u) \cdot w$ is the volume of this same parallelepiped. Also the dot product is commutative.

d) false; the dot product is not defined ~~for~~ between a vector (u) and scalar $(v \cdot w)$.

False c) $u \cdot (v \times w)$ is the tripple product so

$$u \cdot (v \times w) = \det \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$(v \times u) \cdot w = w \cdot (v \times u) = \det \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = - \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

1DU

$u \cdot (v \times w)$

$(v \times u) \cdot w$

5. Use the cross product to find a parametrization for the line of intersection of the planes described by the formulas

$$2x + 3y + z = 1$$
$$x + y + z = 2$$

10 pts

$$\begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (3-1)\hat{i} - (2-1)\hat{j} + (2-3)\hat{k}$$

$$= 2\hat{i} - \hat{j} - \hat{k}$$

$y = 0$

$-2 + 3 = 1$
$-1 + 3 = 2$

$2x + z = 1$
$x + z = 2$

point in both planes: $(-1, 0, 3)$

$2x + z = 1$
$-x - z = -2$

$x = -1$

$z = 3$

parametrization: $x = -1 + 2t$

$y = 0 - t$

$z = 3 - t$

10 ✓

6. Find the distance from the point $\cancel{P}(1,1,1)$ to the plane described by $x+3y+2z=1$. 10 pts

S (1,1,1)

$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$

P(1,0,0)

$\vec{PS} = \langle 1,1,1 \rangle - \langle 1,0,0 \rangle$

$= \langle 0,1,1 \rangle$

$\vec{n} = \langle 1,3,2 \rangle$

$|\vec{n}| = \sqrt{1+9+4}$

$= \sqrt{14}$

$d = \left| \frac{\langle 0,1,1 \rangle \cdot \langle 1,3,2 \rangle}{\sqrt{14}} \right| = \left| \frac{0+3+2}{\sqrt{14}} \right| = \frac{5}{\sqrt{14}}$

' D O

7. Let γ be the curve described by the vector function $\mathbf{r}(t)$ below. Find parametric equations for the tangent line to this curve at the point $P(3,2,1)$. $(t=1)$

$$\mathbf{r}(t) = 3t^2\mathbf{i} + (t^3+1)\mathbf{j} + t\mathbf{k}$$

10 pts

$r'(t) = 6t\hat{i} + (3t^2)\hat{j} + \hat{k}$

$r'(1) = 6\hat{i} + 3\hat{j} + \hat{k}$

parametrization: $x = 3 + 6t$

$y = 2 + 3t$

$z = 1 + t$

100

8. Let γ be the curve described by the vector function $\mathbf{r}(t)$ below. Give an integral that expresses the arc length of the segment of the curve traced out by $\mathbf{r}(t)$ for $2 \le t \le 5$. **Do not attempt to evaluate the integral!**

$$\mathbf{r}(t) = 3t^2\mathbf{i} + (t^3+1)\mathbf{j} + t\mathbf{k}$$

10 pts

$r'(t) = v(t) = 6t\hat{i} + 3t^2\hat{j} + \hat{k}$

$|v(t)| = \sqrt{36t^2 + 9t^4 + 1}$

arc length $= \int_2^5 \sqrt{36t^2+9t^4+1}\,dt$

100

$(\frac{x}{2})^2 + (\frac{y}{4})^2 = 1$

9. Let γ be the intersection of the cylinder $\frac{x^2}{4} + \frac{y^2}{16} = 1$ and the plane $z = x + y$. You plan to use the formula for curvature on the last page of the test to compute the curvature of γ as a function of t. Give explicit expressions for the vector functions $\mathbf{v}(t)$ and $\mathbf{a}(t)$ which you would substitute into the formula for the curvature. **Do** not **carry the calculation further.** 10 pts

$x = 2\cos t \quad y = 4\sin t \quad z = 2\cos t + 4\sin t$

use $\kappa = \frac{|v \times a|}{|v|^3}$ to calculate curvature function

$$\begin{cases} v(t) = (-2\sin t)\hat{i} + (4\cos t)\hat{j} + (-2\sin t + 4\cos t)\hat{k} \\ a(t) = (-2\cos t)\hat{i} + (-4\sin t)\hat{j} + (-2\cos t - 4\sin t)\hat{k} \end{cases}$$

$$v \times a = \begin{vmatrix} i & j & k \\ -2\sin t & 4\cos t & -2\sin t + 4\cos t \\ -2\cos t & -4\sin t & -2\cos t - 4\sin t \end{vmatrix}$$

$$= [4\cos t(-2\cos t - 4\sin t) - (-4\sin t)(-2\sin t + 4\cos t)]\hat{i}$$

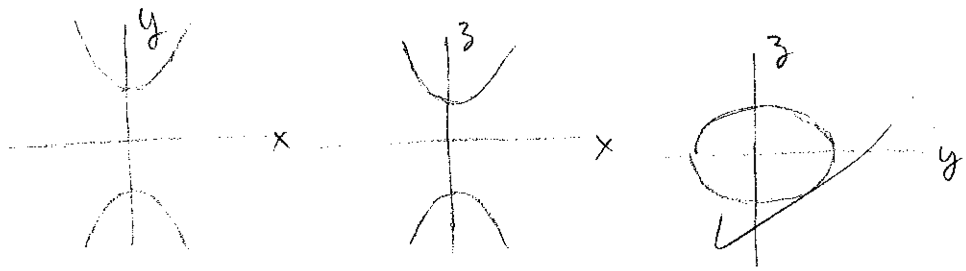
$$- [(-2\sin t)(-2\cos t - 4\sin t) - (-2\cos t)(-2\sin t + 4\cos t)]\hat{j}$$

$$+ [(-2\sin t)(-4\sin t) - (-2\cos t)(4\cos t)]\hat{k}$$

$z=0 \quad \frac{y^2}{4}-\frac{x^2}{9}=4 \qquad x=0 \quad z^2+\frac{y^2}{4}=4$

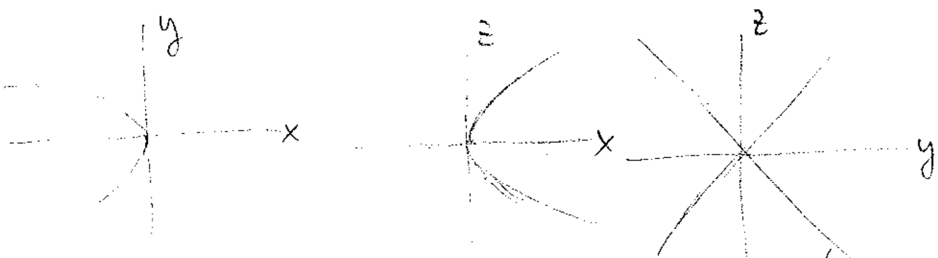
$y=0 \quad z^2-\frac{x^2}{9}=4 \qquad \frac{z^2}{4}+\frac{y^2}{16}=1$

10. The surface described by $z^2+\frac{y^2}{4}=4+\frac{x^2}{9}$ would look most like which of the pictures (a)-(l) indicated on the last page of the test. 6 pts



most like I)

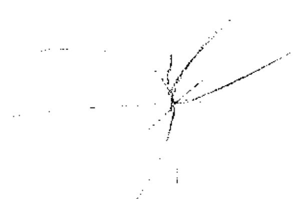
11. The surface described by $x+y^2=z^2$ would look most like which of the pictures (a)-(l) indicated on the last page of the test. 6 pts



$x=0 \quad y^2=z^2 \qquad y=0 \quad x=z^2 \qquad z=0 \quad x+y^2=0$

$x=-y^2$

most like K)



12. Calculate the tangential and normal components of acceleration as a function of t for 10 pts

$$\mathbf{r}(t) = 3t^2\mathbf{i} + (t^3 + 1)\mathbf{j} + \cancel{4}\, t\hat{k}$$

$v(t) = 6t\hat{i} + 3t^2\hat{j} + \hat{k}$ $a(t) = 6\hat{i} + 6t\hat{j}$

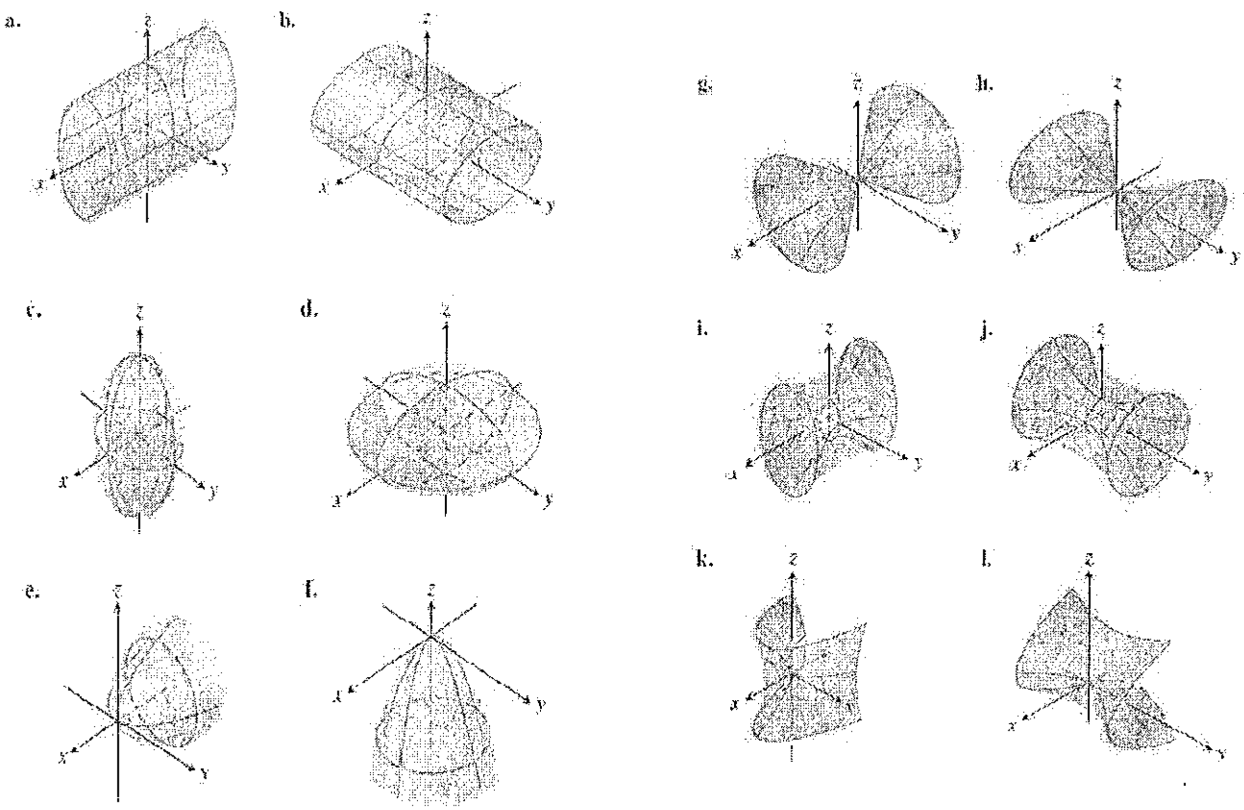
$|v(t)| = \sqrt{36t^2 + 9t^4 + 4}$ $|a|^2 = 36 + 36t^2$

$a_T = |v(t)|' = \frac{1}{2}(36t^2 + 9t^4 + 1)^{-\frac{1}{2}}(72t + 36t^3)$

(differentiate $|v(t)| = \sqrt{36t^2 + 9t^4 + 1}$ with respect to t)

$a_N = \sqrt{|\vec{a}|^2 - a_T^2}$

$= \sqrt{\underbrace{36 + 36t^2}_{|\vec{a}|^2} - \underbrace{\left[\frac{1}{2}(36t^2 + 9t^4 + 1)^{-\frac{1}{2}}(72t + 36t^3)\right]^2}_{a_T^2}}$



Formulas for Curves in Space

Unit tangent vector:	$\mathbf{T} = \dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$
Principal unit normal vector:	$\mathbf{N} = \dfrac{d\mathbf{T}/dt}{\lvert d\mathbf{T}/dt\rvert}$
Binormal vector:	$\mathbf{B} = \mathbf{T} \times \mathbf{N}$
Curvature:	$\kappa = \left\lvert \dfrac{d\mathbf{T}}{ds} \right\rvert = \dfrac{\lvert\mathbf{v} \times \mathbf{a}\rvert}{\lvert\mathbf{v}\rvert^3}$
Torsion:	$\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{\lvert\mathbf{v} \times \mathbf{a}\rvert^2}$
Tangential and normal scalar components of acceleration:	$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ $a_T = \dfrac{d}{dt}\lvert\mathbf{v}\rvert$ $a_N = \kappa\lvert\mathbf{v}\rvert^2 = \sqrt{\lvert\mathbf{a}\rvert^2 - a_T^{\,2}}$