, 100

1. Either find the following limit or prove that it does not exist. 10 pts

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x+y+z}{2x+3y-z}$$

$\lim_{\substack{(x,y,z)\to(0,0,0) \\ \text{along } x\text{-axis}}} \frac{x+y+z}{2x+3y-z} = \frac{x}{2x} = \frac{1}{2}$

$\lim_{\substack{(x,y,z)\to(0,0,0) \\ \text{along } y\text{-axis}}} \frac{x+y+z}{2x+3y-z} = \frac{y}{3y} = \frac{1}{3}$

$\lim_{\substack{(x,y,z)\to(0,0,0) \\ \text{along } z\text{-axis}}} \frac{x+y+z}{2x+3y-z} = -1$

they are not equal, so

the limit doesn't exist

$1 \textcircled{0}$

2. Let
$$f(x,y) = \frac{x^4}{x^2+y^2}.$$
Clearly $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Illustrate this by finding a $\delta > 0$ such that 10 pts
$$\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y)| < \frac{1}{4}.$$

$$\left|\frac{x^4}{x^2+y^2}\right| = \left|\frac{x^2 \cdot x^2}{x^2+y^2}\right| = |x^2| \cdot \left|\frac{x^2}{x^2+y^2}\right|$$
$$\leq |x^2|$$
$$\leq |x^2+y^2| < \frac{1}{4}$$
$$\because \delta > 0 \qquad \therefore \delta^2 = \frac{1}{4}$$
$$\therefore \text{let } \delta = \frac{1}{2}$$

$\therefore$ let $\delta = \frac{1}{2}$, $\sqrt{x^2+y^2} < \frac{1}{2} \implies$ $\left|\frac{x^4}{x^2+y^2}\right| = x^2\left|\frac{x^2}{x^2+y^2}\right| \leq |x^2|$
$$\leq x^2+y^2 = (\sqrt{x^2+y^2})^2$$
$$< \frac{1}{4}$$

100

3. Find an equation for the tangent plane to the surface $2z^2 - x^2e^{3y} = 1$ at the point $(1, 0, 1)$. 10 pts

$f = 2z^2 - x^2e^{3y} - 1 = 0$

$\nabla f = (-2xe^{3y})i - (3x^2e^{3y})j + (4z)k$

$\nabla f\big|_{(1,0,1)} = -2i - 3j + 4k$

$-2(x-1) - 3(y) + 4(z-1) = 0$

$-2x + 2 - 3y + 4z - 4 = 0$

$-2x - 3y + 4z - 2 = 0$

100

4. Find the linearization $L(x, y)$ of the function $f(x, y) = \sin x e^{-3y}$ at the point $(0, 0)$. Then find an upper bound on the error for the approximation $f(x, y) \approx L(x, y)$ on the rectangle R defined by $|x| \le .1$, $|y| \le .1$. Use the observations that on R, $e^{-3y} \le e^{.3} \le 2$, $|\sin x| \le 1$, and $|\cos x| \le 1$. 10 pts

$$L(x,y) = \underset{0}{f(0,0)} + \underset{1}{f_{x(0,0)}}(x-x_0) + \underset{0}{f_{y(0,0)}}(y-y_0)$$

$$f(0,0) = 0$$

$$f_x = \cos x e^{-3y} \quad f_x(0,0) = 1$$

$$f_y = -3\sin x e^{-3y} \quad f_y(0,0) = 0.$$

$$\therefore L(x,y) = 0 + (x-0) = x$$

$$f_{xx} = -\sin x e^{-3y} \quad \le |\sin x| \cdot |e^{-3y}| = 2$$

$$f_{yy} = 9\sin x e^{-3y} \quad \le 9|\sin x||e^{-3y}| = \boxed{18}$$

$$f_{xy} = -3\cos x e^{-3y} \quad \le 3|\cos x||e^{-3y}| = 6$$

$$\therefore M = 18.$$

$$\therefore |E(x,y)| \le \frac{1}{2} \times 18 \times (0.1 + 0.1)^2 \qquad 0.2$$

$$= 9 \times 0.04$$

$$= 0.36$$

100

5. Use Taylor series to find a quadratic approximation to $f(x,y) = (\sin x)e^{-3y}$ at $(0,0)$. (Order 2 Taylor approximation.) 10 pts

$$f(x,y) \approx \underset{0}{f(0,0)} + x\underset{1}{f_x} + y\underset{0}{f_y} + \frac{1}{2}(x^2\underset{0}{f_{xx}} + 2xy\underset{-3}{f_{xy}} + y^2\underset{0}{f_{yy}}$$

$f(0,0) = 0$

$f_x = \cos x \cdot e^{-3y}$, $f_x(0,0) = 1$

$f_y = -3(\sin x) \cdot e^{-3y}$, $f_y(0,0) = 0$

$f_{xx} = -\sin x \cdot e^{-3y}$, $f_{xx}(0,0) = 0$

$f_{yy} = 9 \sin x \, e^{-3y}$ $f_{yy}(0,0) = 0$

$f_{xy} = -3\cos x \, e^{-3y}$ $f_{xy}(0,0) = -3$

$$\therefore f(x,y) \approx 0 + x + \cancel{y0} + \frac{1}{2}(\cancel{x^2 0} + 2xy \cdot (-3) + \cancel{y^2 0})$$

$$= x - 3xy$$

100

6. Find all critical points for

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

and identify each as either a local maximum, local minimum, or saddle point. 10 pts

$$\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 = y \\ y^2 = x \end{cases} \Rightarrow y^4 - y = 0$$

$$y(y^3 - 1) = 0$$

$\therefore y = 0$ or $y = 1$

$\therefore$ critical points $(0,0)$ and $(1,1)$ $\therefore$ $x = 0$ or $x = 1$

$f_{xx} = 6x$, $f_{yy} = 6y$ $f_{xy} = -3$

when $(0,0)$.

$f_{xx} = 0$, $f_{yy} = 0$, $f_{xx}f_{yy} - f_{xy}^2 = -9 < 0$.

$\therefore$ $(0,0)$ is a saddle point.

when $(1,1)$.

$f_{xy} = 6 > 0$ $f_{yy} = 6$, $f_{xy} = -3$

$f_{xx}f_{yy} - f_{xy}^2 = 36 - 9 = 27 > 0$

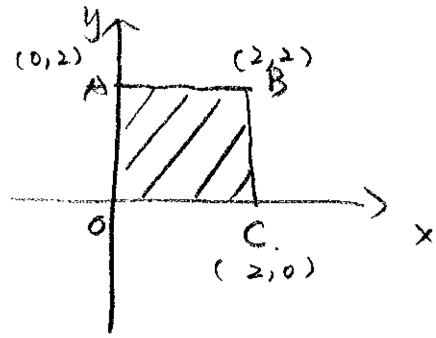
$f_{xx} > 0$

$\therefore$ $(1,1)$ is a local minimum point,

7. Find the absolute maximum and the absolute minimum of the function $f(x, y)$ from Problem 6 over the region defined by $0 \le x \le 2, 0 \le y \le 2$.

10 pts

$f(x,y) = 4 + x^3 + y^3 - 3xy$



① on OA:

$f(0,y) = 4 + y^3$

$O: f(0,0) = \boxed{4}$; $A: f(0,2) = \boxed{12}$

$f'(0,y) = 3y^2 = 0$

$\therefore y=0, x=0$ isn't a interior point

② on OC:

$f(x,0) = 4 + x^3$

$O: f(0,0) = \boxed{4}$; $C: f(2,0) = \boxed{12}$

$f'(x,0) = 3x^2 = 0$, $(0,0)$ isn't a interior point.

③ on AB:

$f(x,2) = 4 + x^3 + 8 - 6x = x^3 - 6x + 12$.

$A: f(0,2) = \boxed{12}$; $B: f(2,2) = \boxed{8}$

$f'(x,2) = 3x^2 - 6 = 0$ $\quad x(x-2) = 0$

$\therefore (0,0)$ and $(2,2)$ they aren't interior points.

④ on BC: $f(2,y) = 4 + 8 + y^3 - 6y$

$B: f(2,2) = \boxed{8}$; $C: f(2,0) = \boxed{12}$

$f'(2,y) = 3y^2 - 6y = 0$, $y(y-2) = 0$.

$\therefore (0,0)$ and $(2,2)$ they aren't interior points.

⑤ for interior points

$$\begin{cases} f_x = 3x^2 - 3y = 0 \\ f_y = 3y^2 - 3x = 0 \end{cases} \Rightarrow (0,0), (1,1); \ (0,0) \text{ isn't an interior point}$$

$f(1,1) = 4 + 1 + 1 - 3 = \boxed{3}$

$\therefore$ the absolute minimum is 3 at point $(1,1)$

and the absolute maximum is 12 at point $(0,2)$ and $(2,0)$

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & u, v & \end{tikzcd}$$

8. Suppose that $z = f(x, y)$ where $\frac{\partial f}{\partial x}(x, y) = 3x$ and $\frac{\partial f}{\partial y}(x, y) = xe^y$. Suppose also that $x = e^{3u}v$, $y = e^{2u}\sin v$. Find $\frac{\partial z}{\partial v}$ at the point $(u, v) = (0, \pi)$. 10 pts

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = 3x, \quad \frac{\partial z}{\partial y} = x \cdot e^y$$

$$\frac{\partial x}{\partial v} = e^{3u}, \quad \frac{\partial y}{\partial v} = e^{2u} \cdot \cos v$$

at point $(0, \pi) = (u, v)$

$$\frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial v} = 1 \cdot (-1) = -1$$

$$x = e^{3u} v = \pi, \quad y = e^{2u} \sin v = 0$$

$$\therefore \frac{\partial z}{\partial v} = (3\pi)(1) + (\pi \cdot e^0) \cdot (-1)$$

$$= 3\pi - \pi = 2\pi$$

100

9. Find the directional derivative of the function $f(x, y) = x^2+y^3-3xy+5$ at the point $(0, 1)$ in the direction $< 3, 4 >$. 4 pts

$$\nabla f = (2x-3y)i + (3y-3x)j$$

$$\nabla f \big|_{(0,1)} = -3i + 3j.$$

$$u = \frac{3}{\sqrt{9+16}} i + \frac{4}{\sqrt{9+16}} j = \frac{3}{5} i + \frac{4}{5} j$$

$$(d_u f)\big|_{(0,1)} = \nabla f \cdot u = (-3) \times \frac{3}{5} + 3 \times \frac{4}{5}$$

$$= -\frac{9}{5} + \frac{12}{5} = \frac{3}{5}$$

10. Find the direction in which the function from Problem 9 is decreasing most rapidly at the point $(0, 1)$. 3 pts

$$f(x,y) = x^2 + y^3 - 3xy + 5$$

$$\nabla f = (2x-3y) i + (3y^2 - 3x) j$$

$$\nabla f \big|_{(0,1)} = -3i + 3j \quad |\nabla f| = \sqrt{9+9} = 3\sqrt{2}$$

$$-u = -\frac{\nabla f}{|\nabla f|} = -\left(-\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j\right)$$

$$= \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

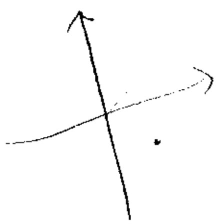
3.5

97

- 3

11. Suppose that the vector **u** makes a 60° angle with respect to the direction found in Problem 10. What is the directional derivative of the function $f(x, y)$ from Problem 9 at the point $(0, 1)$ in the direction of **u**. 3 pts

$D_u f$



$$(D_u f)\big|_{(0,1)} = \nabla f \cdot u$$

$$= (-3i + 3j) \cdot u = ?$$

$$\nabla f \cdot u = |\nabla f||u| \cos\theta$$

$$= \cancel{\ \ }$$

$$= |\langle 3, -3\rangle| \tfrac{1}{2} = \frac{\sqrt{18}}{2}$$

12. If we use the method of Legrange multipliers to find the minimum of $f(x,y) = xy$ subject to the constraint $x^2+y^2=1$, what are the possible values for the Legrange multiplier λ? *I only want the value of λ. Do not carry the solution further.* 10 pts

$f(x,y) = xy$, $g = x^2+y^2-1$

$\nabla f = y\,i + x\,j$. $\nabla g = (2x)\,i + (2y)\,j$.

$$\begin{cases} y = 2x \cdot \lambda \\ x = 2y \cdot \lambda \\ x^2+y^2 = 1 \end{cases}$$

$y = 2(2y\lambda)\cdot\lambda$

$y = 4y\cdot\lambda^2$

① $y=0$, $x=0$ not on $x^2+y^2=1$

② $y \neq 0$, $\lambda^2 = \frac{1}{4}$ $\therefore$ $\lambda = \pm\frac{1}{2}$