MA 174 EXAM I 02/18/2011

Name: Solution PUID: __________ Section: __________

SHOW ALL YOUR WORK. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED.

Points awarded

1. (7 points) E	5. (24 points)
2. (7 points) B	6. (12 points)
3. (7 points) B	7. (12 points)
4. (16 points)	8. (15 points)

Total Points: __________

1. (7 points) If

$$L = \lim_{(x,y,z)\to(0,0,0)} \frac{x-y-2z}{x+y+z},$$

then

A. $L = 1$
B. $L = -1$
C. $L = -2$
D. $L = 0$
$\boxed{\text{E.}}$ the limit does not exist

along x-axis $L=1$
y-axis $L=-1$
z-axis $L=-2$

2. (7 points) The surface defined by $z^2 = 4x^2 + 9y^2$ is a

A. hyperbolic paraboloid
$\boxed{\text{B.}}$ elliptical cone
C. elliptical paraboloid
D. ellipsoid
E. hyperboloid

3. (7 points) Which of the following statements are true for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$?

(i) if $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
(ii) if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
(iii) $\mathbf{u} \cdot \mathrm{Proj}_{\mathbf{u}}\mathbf{v} = 0$
(iv) $\mathbf{u} \times \mathrm{Proj}_{\mathbf{u}}\mathbf{v} = \mathbf{0}$

A. (i) and (iii) only
$\boxed{\text{B.}}$ (i) and (iv) only
C. (ii) and (iii) only
D. (ii) and (iv) only
E. all are true

4. (a)(10 points) Find the plane determined by the lines $x = t,\ y = -t + 2,\ z = t + 1$ and $x = 2s + 2,\ y = s + 3,\ z = 5s + 6$.

Sol: The intersection of the two lines is point $P(0,2,1)$

OR $P(0,2,1)$ lies on the lines, Then it is also on the plane.

normal of the plane is $\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\vec{i} - 3\vec{j} + 3\vec{k}$

Then the plane is

$$-6x - 3y + 3z = -6(0) - 3(2) + 3(1) = -3$$

$\Rightarrow$ $\boxed{2x + y - z = 1}$

Note, you may pick any point on any of the lines. for instance. pick $P(1,1,2)$ $(t=1)$

(b)(6 points) Find the distance from point $S(3, 3, 2)$ to the plane in (a).

Sol: $\vec{PS} = \langle 3-0,\ 3-2,\ 2-1 \rangle = \langle 3, 1, 1 \rangle$

$$d = \frac{|\vec{PS} \cdot \vec{n}|}{|\vec{n}|} = \frac{|\langle 3,1,1\rangle \cdot \langle 2,1,-1\rangle|}{\sqrt{2^2 + 1^2 + (-1)^2}} = \frac{6}{\sqrt{6}} = \boxed{\sqrt{6}}$$

5. Let

$$\mathbf{r}(t) = (2+t)\ \mathbf{i} + (t^2+3)\ \mathbf{j} + (\frac{2}{3}t^3 - 5)\ \mathbf{k}.$$

(a)(8 points) Find the parametric equation for the tangent line to the curve $\mathbf{r}(t)$ at $t = 3$.

Sol: $\vec{r}(3) = 5\vec{i} + 12\vec{j} + 13\vec{k}$ point $(5, 12, 13)$

$\vec{v}(t) = \vec{i} + 2t\vec{j} + 2t^2\vec{k}$

$\vec{v}(3) = \vec{i} + 6\vec{j} + 18\vec{k}$ $\vec{v} = \langle 1, 6, 18\rangle$

the equations for the line are
$$\boxed{\begin{array}{l} x = 5+t \\ y = 12+6t \qquad -\infty < t < \infty \\ z = 13+18t \end{array}}$$

(b)(8 points) Find the arc length of $\mathbf{r}(t)$ from $t = 0$ to $t = 3$.

Sol: $\vec{v}(t) = \vec{i} + 2t\vec{j} + 2t^2\vec{k}$

$|\vec{v}| = \sqrt{1 + (2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 4t^4} = 2t^2 + 1$

$S = \int_0^3 |\vec{v}|\, dt = \int_0^3 (2t^2+1)\, dt = \frac{2}{3}t^3 + t \Big]_0^3 = 18 + 3 = \boxed{21}$

(c)(8 points) Calculate the tangential and normal components of the acceleration.

Sol: $\vec{a}(t) = 2\vec{j} + 4t\vec{k}$

$|\vec{a}|^2 = 4 + 16t^2$

$$\boxed{\begin{array}{l} a_T = \dfrac{d|\vec{v}|}{dt} = \dfrac{d}{dt}(2t^2+1) = 4t \\ a_N = \sqrt{|\vec{a}|^2 - a_T^2} = 2 \end{array}}$$

6. (12 points) A particle starts at the origin with initial velocity $\mathbf{i} - \mathbf{j} + \frac{2}{3}\mathbf{k}$. Its acceleration is $2\,\mathbf{i} + 4\,\mathbf{j} + 2t\,\mathbf{k}$. Find its position at $t = 1$.

Sol: $\vec{a}(t) = 2\vec{i} + 4\vec{j} + 2t\vec{k}$

$$\Rightarrow \vec{v}(t) = \int (2\vec{i} + 4\vec{j} + 2t\vec{k})\,dt = 2t\vec{i} + 4t\vec{j} + t^2\vec{k} + \vec{C_1}$$

$$\vec{v}(0) = \vec{i} - \vec{j} + \frac{2}{3}\vec{k}$$

$$\Rightarrow \vec{C_1} = \vec{i} - \vec{j} + \frac{2}{3}\vec{k}$$

$$\Rightarrow \vec{v}(t) = (2t+1)\vec{i} + (4t-1)\vec{j} + (t^2 + \frac{2}{3})\vec{k}$$

$$\Rightarrow \vec{r}(t) = \int \left[(2t+1)\vec{i} + (4t-1)\vec{j} + (t^2 + \frac{2}{3})\vec{k}\right] dt$$

$$= (t^2 + t)\vec{i} + (2t^2 - t)\vec{j} + (\frac{1}{3}t^3 + \frac{2}{3}t)\vec{k} + \vec{C_2}$$

$$\vec{r}(0) = 0 \quad \Rightarrow \vec{C_2} = \vec{0}$$

$$\Rightarrow \vec{r}(t) = (t^2 + t)\vec{i} + (2t^2 - t)\vec{j} + (\frac{1}{3}t^3 + \frac{2}{3}t)\vec{k}$$

$$\Rightarrow \boxed{\vec{r}(1) = 2\vec{i} + \vec{j} + \vec{k}}$$

7. (12 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1,1,1)$ if $z(x,y)$ is defined by the equation

$$z^3 - 2xy - yz - y^3 + 3 = 0.$$

Sol:

$$\frac{\partial}{\partial z}(z^3 - 2xy - yz - y^3 + 3) = 0$$

$$\Rightarrow \quad 3z^2 \frac{\partial z}{\partial x} - 2y - y\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{2y}{3z^2 - y} \quad \Rightarrow \quad \boxed{\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{2}{3-1} = \frac{2}{2} = 1}$$

$$\frac{\partial}{\partial y}(z^3 - 2xy - yz - y^3 + 3) = 0$$

$$\Rightarrow \quad 3z^2\frac{\partial z}{\partial y} - 2x - z - y\frac{\partial z}{\partial y} - 3y^2 = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{2x + z + 3y^2}{3z^2 - y} \quad \Rightarrow \quad \boxed{\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = \frac{2+1+3}{3-1} = 3}$$

8. (15 points) Let C be the intersection of $x^2 + y^2 = 4$ and $z = 5$, find the curvature and torsion of C at $(2, 0, 5)$.

Sol: ① $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 5\vec{k}$ $\boxed{(2,0,5) \Rightarrow t=0}$

$$\vec{v}(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} \Rightarrow |\vec{v}| = 2$$

$$\vec{a}(t) = -2\cos t\,\vec{i} - 2\sin t\,\vec{j}$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & 2\cos t & 0 \\ -2\cos t & -2\sin t & 0 \end{vmatrix} = 4\vec{k} \Rightarrow |\vec{v}\times\vec{a}| = 4$$

$$\Rightarrow \kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{4}{2^3} = \boxed{\frac{1}{2}}$$

$$\tau = \frac{\begin{vmatrix} \vec{v} \\ \vec{a} \\ \vec{a}' \end{vmatrix}}{|\vec{v}\times\vec{a}|^2} = \frac{\begin{vmatrix} -2\sin t & 2\cos t & 0 \\ -2\cos t & -2\sin t & 0 \\ 2\sin t & -2\cos t & 0 \end{vmatrix}}{16} = \frac{0}{16} = \boxed{0}$$

② $$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{-2\sin t\,\vec{i} + 2\cos t\,\vec{j}}{2} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j} \Rightarrow \left|\frac{d\vec{T}}{dt}\right| = 1$$

$$\Rightarrow \kappa = \frac{|d\vec{T}/dt|}{|\vec{v}|} = \boxed{\frac{1}{2}}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \vec{k} \Rightarrow \frac{d\vec{B}}{dt} = \vec{0} \Rightarrow \frac{d\vec{B}}{dt}\cdot\vec{N} = 0$$

$$\Rightarrow \tau = -\frac{\frac{d\vec{B}}{dt}\cdot\vec{N}}{|\vec{v}|} = -\frac{0}{2} = \boxed{0}$$