MA 174 EXAM II 04/01/2011

Name: Solution PUID: ____________ Section: ____________

SHOW ALL YOUR WORK. NO CALCULATORS, BOOKS, OR PAPERS ARE ALLOWED.

Points awarded

1. (10 points) ____________
2. (10 points) ____________
3. (10 points) ____________
4. (10 points) ____________
5. (15 points) ____________
6. (10 points) ____________
7. (10 points) ____________
8. (10 points) ____________
9. (15 points) ____________

Total Points: ________________

1. (10 points) Find an equation for the tangent plane of the surface

$$xe^y + \cos(xy) + y - z^2 = ln2$$

at point $(0, ln2, 1)$.

Sol: let $F(x,y,z) = xe^y + \cos xy + y - z^2 - \ln 2 = 0$

$F_x = e^y - y\sin xy$ $\qquad F_x|_{(0,\ln 2,1)} = e^{\ln 2} - \ln 2 \sin 0 = 2$

$F_y = xe^y - x\sin xy + 1$ $\qquad F_y|_{(0,\ln 2,1)} = 0 - 0 + 1 = 1$

$F_z = -2z$ $\qquad F_z|_{(0,\ln 2,1)} = -2$

tangent plane is

$$2(x-0) + 1(y - \ln 2) - 2(z-1) = 0$$

or $\boxed{2x + y - 2z = \ln 2 - 2}$

2. (10 points) Find $(\frac{\partial w}{\partial x})_y$ at point $(x, y, z) = (0, 1, \pi)$ if
$w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0.$

Sol:

$$\left(\frac{\partial w}{\partial x}\right)_y = 2x + 2z\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad (\text{let } F(x,y,z) = y\sin z + z\sin x)$$

$$= -\frac{z\cos x}{y\cos z + \sin x} \Rightarrow \frac{\partial z}{\partial x}\Big|_{(0,1,\pi)} = -\frac{\pi\cos 0}{\cos\pi + \sin 0} = \pi$$

$$\left(\frac{\partial w}{\partial x}\right)_y\Big|_{(0,1,\pi)} = 2(0) + 2\pi\cdot\pi = \boxed{2\pi^2}$$

3. (10 points) Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^{2x-y}$ at $(0, 0)$.

Sol: $P_2(x,y) = f(0,0) + x f_x + y f_y + \frac{1}{2}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})$

$f(0,0) = e^0 = 1$

$f_x(x,y) = 2e^{2x-y} = 2$ at $(0,0)$

$f_y = -e^{2x-y} = -1$ at $(0,0)$

$$\left.\begin{aligned} f_{xx} &= 4e^{2x-y} = 4 \\ f_{yy} &= e^{2x-y} = 1 \\ f_{xy} &= -2e^{2x-y} = -2 \end{aligned}\right\} \text{at } (0,0)$$

$$\boxed{P_2(x,y) = 1 + (2x-y) + \frac{1}{2}(4x^2 - 4xy + y^2)}$$

4. (10 points) If the derivative of $f(x, y)$ at a point P in the direction of $\mathbf{i}+\mathbf{j}$ is $3\sqrt{2}$ and in the direction of $\mathbf{i}-\mathbf{j}$ is $2\sqrt{2}$, what is the gradient of $f(x, y)$ at the point P?

Sol: let $\vec{\nabla} f\big|_P = a\vec{i} + b\vec{j}$

$D_u f\big|_{P_0} = \vec{\nabla} f\big|_{P_0} \cdot \vec{u}$ — $\vec{u}$ is a unit vector $\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$

$$\left.\begin{aligned} (a\vec{i} + b\vec{j}) \cdot \left(\frac{\vec{i}+\vec{j}}{\|\vec{i}+\vec{j}\|}\right) = 3\sqrt{2} &\Rightarrow a+b=6 \\ (a\vec{i} + b\vec{j}) \cdot \left(\frac{\vec{i}-\vec{j}}{\|\vec{i}-\vec{j}\|}\right) = 2\sqrt{2} &\Rightarrow a-b=4 \end{aligned}\right\}$$

$\Rightarrow a = 5 \quad b = 1$

Thus $\boxed{\vec{\nabla} f\big|_{P_0} = 5\vec{i} + \vec{j}}$

5. (15 points) Find all critical points of function

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

and identify each as a local maximum, local minimum or saddle point.

Sol: $f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3$

$f_y = 4y^3 - 4x = 0$

$\Rightarrow x^9 - x = 0$

$\Rightarrow x(x^4+1)(x^2+1)(x^2-1) = 0$

$\Rightarrow x = 0 \quad x = 1 \quad \text{or} \quad x = -1$

$y = x^3 \Rightarrow y = 0 \quad y = 1 \quad \text{or} \quad y = -1$

critical points are $(0,0)$, $(1,1)$ & $(-1,-1)$

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

$H = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$

$H(0.0) = -16 < 0 \Rightarrow (0.0)$ is a saddle point

$H(1.1) = 144 - 16 > 0$ & $f_{xx}(1.1) = 12 > 0 \Rightarrow (1.1)$ — local MIN

$H(-1.-1) = 144 - 16 > 0$ & $f_{xx}(-1.-1) = 12 > 0 \Rightarrow (-1.-1)$ local MIN

(0.0) — saddle point

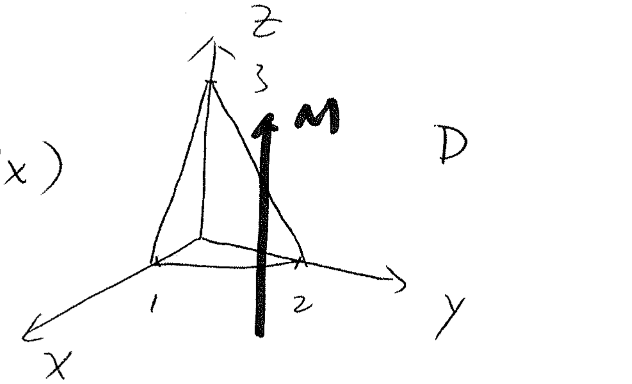
(1.1) & $(-1,-1)$ — local MINIMUM

Answer may vary $\quad V = \int_0^2 \int_0^{1-\frac{1}{2}y} \int_0^{3-3x-\frac{3}{2}y} dz\,dx\,dy$

6. (10 points) Set up an iterated triple integral that gives the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$. Do NOT evaluate the integral.

Sol: The plane passing through $(1,0,0)$, $(0,2,0)$ & $(0,0,3)$ is

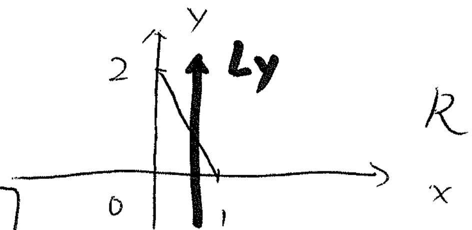
$$6x+3y+2z=6$$



use z-simple (order $dz\,dy\,dx$)

z-limits are 0 and $\dfrac{6-6x-3y}{2}$

The line on xy-plane is $6x+3y=6$ or $2x+y=2$



y-limits are 0 and $2-2x$

x-limits are 0 and 1

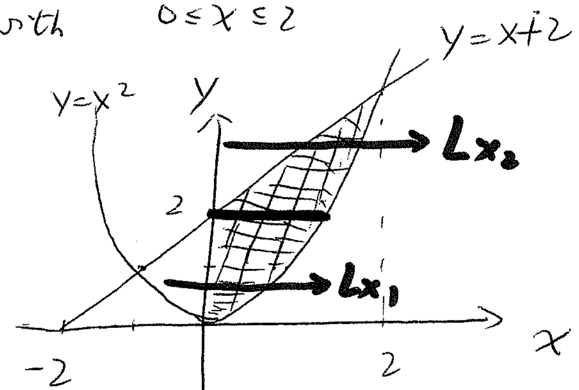
$$\boxed{V = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-\frac{3}{2}y} dz\,dy\,dx}$$

7. (10 points) Sketch the region of the integration for the integral $\int_0^2 \int_{x^2}^{x+2} f(x,y)dydx$ and write an equivalent integral with the order of integration reversed.

Sol: the boundary of the region is

$y=x+2 \qquad y=x^2 \qquad$ with $\quad 0 \le x \le 2$

R is the shaded region



use x-simple, divide the region into 2 parts by the line $y=2$

Lx_1: x-limits are 0 & $\sqrt{y}$

y-limits are 0 & 2

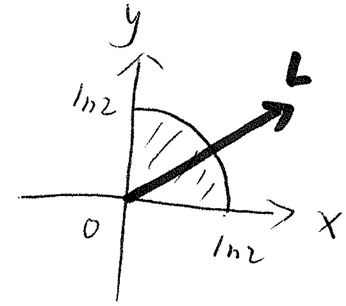
Lx_2: x-limits are $y-2$ & $\sqrt{y}$

y-limits are 2 & 4

$$\boxed{\int_0^2 \int_{x^2}^{x+2} f(x,y)\,dy\,dx = \int_0^2 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy + \int_2^4 \int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy}$$

8. (10 points) Evaluate $\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy.$

Sol: the region of integration is the circle $x^2+y^2=(\ln 2)^2$ in the 1^{st} quadrant.



change the double integral into the one in polar coordinates:

$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$$\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\frac{\pi}{2}}\underbrace{\int_0^{\ln 2} e^r\cdot r\,dr}_{\text{integration by parts}}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} (e^r\cdot r - e^r)\Big]_0^{\ln 2}\,d\theta = \left(e^{\ln 2}\ln 2 - e^{\ln 2}+1\right)\frac{\pi}{2} = \boxed{\frac{(2\ln 2-1)\pi}{2}}$$

9. (15 points) Find the greatest and smallest values that the function $f(x,y) = x^2+y^2+xy$ takes on the disc $x^2+y^2 \le 1$.

$$\boxed{\begin{array}{l} f_{min} = 0 \text{ at } (0,0) \\ f_{max} = 3/2 \text{ at } \pm\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\end{array}}$$

Sol: ① interior points for critical points

$f_x = 2x+y = 0$
$f_y = 2y+x = 0$ $\Rightarrow \begin{cases} x=0 \\ y=0\end{cases} \Rightarrow$ $(0,0)$ is a critical pts b/c $0^2+0^2<1$

② boundary $x^2+y^2=1$

let $g(x,y) = x^2+y^2-1 = 0$, use the method of lagrange multipliers

$\vec{\nabla f} = \lambda\vec{\nabla g}$
$g(x,y) = 0$ $\Rightarrow \begin{cases} 2x+y = 2\lambda x \\ 2y+x = 2\lambda y \\ x^2+y^2=1\end{cases}$ Subtract ! $x-y = 2\lambda(x-y)$

$(x-y)(2\lambda-1) = 0$

Case 1 $\left.\begin{array}{l} x=y \\ x^2+y^2=1\end{array}\right\} \Rightarrow x=y=\pm\frac{\sqrt{2}}{2}$

Case 2 $\lambda = \frac{1}{2} \Rightarrow \left.\begin{array}{l} x=-y \\ x^2+y^2=1\end{array}\right\} \Rightarrow x=-y=\pm\frac{\sqrt{2}}{2}$

③ check all the points

$\underline{f(0,0)=0}$ **MIN** $f|(x=y=\pm\frac{\sqrt{2}}{2}) = 1+\frac{1}{2} = \underline{\frac{3}{2}}$ **MAX** $f|(x=-y=\pm\frac{\sqrt{2}}{2}) = 1-\frac{1}{2} = \frac{1}{2}$