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Justify all answers. A correct answer without supporting justification is worth NO credit!

1. Find the characteristic polynomial $p(\lambda)$ for the following matrix A. *You need not simplify your answer, so long as it does not involve determinants. I know that you can multiply polynomials!* *5 pts.*

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 5 & -4 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

$$\begin{vmatrix} 2-\lambda & 0 & 3 \\ 5 & -4-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= 2-\lambda \begin{vmatrix} -4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} + 3 \begin{vmatrix} 5 & -4-\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (2-\lambda)((-4-\lambda)(2-\lambda)) + 3((5)(1))$$

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2. Given that the characteristic polynomial for the matrix A below is $p(\lambda) = (\lambda^2 - 10\lambda + 20)(\lambda - 3)^2$, find a basis for the $\lambda = 3$ eigenspace.

6 pts.

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

$$A - 3\lambda I = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x + 2z + w = 0 \\ y - z = 0 \end{cases}$$

$x = -2z - w$

$y = z$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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3. Let

$$A = \begin{bmatrix} -8 & 4 & -2 \\ -6 & 2 & -3 \\ -8 & 8 & -8 \end{bmatrix}$$

Verify that the vectors X, Y, and Z are eigenvectors for A where *6 pts.*

$$X = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$AX = \begin{bmatrix} -16+12-8 \\ -12+6-12 \\ -16+24-32 \end{bmatrix} = \begin{bmatrix} -12 \\ -18 \\ -24 \end{bmatrix} = -6\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$AY = \begin{bmatrix} -8+4 \\ -6+2 \\ -8+8 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$AZ = \begin{bmatrix} -8+8-4 \\ -6+24-6 \\ -8+16-16 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ -8 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

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4. Let A be as in Problem 3. Find an eigenvector $W = [x, y, z]^t$ for A such that $y = 0$. *3 pts.*

$$W = 2Y - Z = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 1 \\ 2 - 2 \\ 0 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

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5. Let

$$A = \begin{bmatrix} 1 & 6 & -2 \\ -3 & 4 & 0 \\ -8 & 8 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 12 \\ 20 \\ 54 \end{bmatrix}$$

It is given that X_1, X_2, and X_3 are eigenvectors for A corresponding respectively to the stated eigenvalues λ.

$$X = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} (\lambda = 1), \ Y = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} (\lambda = 2), \ Z = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} (\lambda = 3)$$

Find numbers a, b, and c such that

$$A^{100}B = aX + bY + cZ.$$

Hint: Compute $X + 2Y + Z$. *7 pts.*

$$X + 2Y + Z = \begin{bmatrix} 6 \\ 10 \\ 27 \end{bmatrix} = \frac{B}{2}$$

$$A^{100}B = A^{100}(2X + 4Y + 2Z)$$

$$= 2 \cdot 1^{100} X + 4 \cdot 2^{100} Y + 2 \cdot 3^{100} Z$$

$a = 2$ $\quad b = 4 \cdot 2^{100}$ $\quad c = 2 \cdot 3^{100}$

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6. Let A be as below. Given that the characteristic polynomial of A is $p(\lambda) = (\lambda - 3)^2 + 3$, find **explicit** matrices Q and D where D is diagonal such that $A = QDQ^{-1}$. All that is asked for is Q and D. **Do not compute Q^{-1} or QDQ^{-1}.** *Note:* Your answer might involve complex numbers. *7 pts*

$$A = \begin{bmatrix} 3 & -3 \\ 1 & 3 \end{bmatrix}$$

$$P(\lambda) = (\lambda - 3)^2 + 3 = 0$$
$$(\lambda - 3)^2 = -3$$
$$\lambda - 3 = \pm\sqrt{3}i$$
$$\lambda = 3 \pm \sqrt{3}i$$

for $\lambda = 3 + \sqrt{3}i$

$$A - \lambda i = \begin{bmatrix} -\sqrt{3}i & -3 \\ 1 & -\sqrt{3}i \end{bmatrix} \to \begin{bmatrix} \sqrt{3} & -3i \\ 1 & -\sqrt{3}i \end{bmatrix} \to \begin{bmatrix} 1 & -\sqrt{3}i \\ 1 & -\sqrt{3}i \end{bmatrix} \to \begin{bmatrix} 1 & -\sqrt{3}i \\ 0 & 0 \end{bmatrix}$$

$x = \sqrt{3}iy$

$$\begin{bmatrix} \sqrt{3}i \\ 1 \end{bmatrix}$$

for $\lambda = 3 - \sqrt{3}i$

$$A - \lambda i = \begin{bmatrix} \sqrt{3}i & -3 \\ 1 & \sqrt{3}i \end{bmatrix} \to \begin{bmatrix} -\sqrt{3} & -3i \\ 1 & \sqrt{3}i \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{3}i \\ 1 & \sqrt{3}i \end{bmatrix} \to \begin{bmatrix} 1 & \sqrt{3}i \\ 0 & 0 \end{bmatrix}$$

$x = -\sqrt{3}iy$

$$\begin{bmatrix} -\sqrt{3}i \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \sqrt{3}i & -\sqrt{3}i \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3+\sqrt{3}i & 0 \\ 0 & 3-\sqrt{3}i \end{bmatrix}$$

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7. Suppose that A is an $n \times n$ diagonalizable matrix such that the only eigenvalues of A are 1, -1, and 0. Prove that $A^3 = A$. *8 pts.*

Because ~~A~~ is diagonalizable, A can be written as QDQ^{-1} where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

and $\lambda_i = 1, -1$ or 0 for all $i \in [0, n]$

because $1^3 = 1$, $(-1)^3 = (-1)$, $0^3 = 0$, $\therefore \lambda_i^3 = \lambda_i$.

$$D^3 = \begin{bmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

$$A^3 = QD^3Q^{-1} = QDQ^{-1} = A.$$

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8. Prove that there are no values of a and b for which the matrix A below is diagonalizable. *8 pts.*

$$A = \begin{bmatrix} 2 & 1 & 1 & a \\ 0 & 4 & 0 & b \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$P(\lambda) = (2-\lambda)^2(4-\lambda)^2$$

for $\lambda = 2$. $A - \lambda I = \begin{bmatrix} 0 & 1 & 1 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 0 & 1 & 1 & a \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it is rank3, the null space of ~~it~~ is 1 - dimention

for $\lambda = 4$ the null space of it can not be more than 2 - dimention

∴ there are at most 3 independent eigen vectors but A is a 4 × 4 matrix

$3 < 4$

∴ A is not diagonalizable.

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9. We use the ordered basis $\mathcal{B} = \{[1,1,0]^t, [0,1,1]^t, [0,0,1]^t\}$ to define coordinates for $\mathbb{R}^3$. Find the $\mathcal{B}$ coordinate vector for $X = [1,2,3]^t$. *7 pts.*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$C_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$X' = C_{\mathcal{B}} X = \begin{bmatrix} 1 + 0 + 0 \\ -1 + 2 + 0 \\ 1 - 2 + 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

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10. Recall that $\mathcal{P}_n$ be the space of all polynomial functions of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_k \in \mathbb{R}$, $k = 0 \ldots, n$. Let $L : \mathcal{P}_1 \mapsto \mathcal{P}_2$ be the linear transformation defined by

$$L(y) = 2y' + xy$$

We use the ordered basis $\mathcal{B}_1 = \{1, x+1\}$ for the domain and the ordered basis $\mathcal{B}_2 = \{x^2, x, 1\}$ for the target space of L. Find the matrix M that represents L in these bases. *7 pts.*

$$L(1) = 2(1)' + x(1)$$
$$= x$$
$$L(x+1) = 2(x+1)' + x(x+1)$$
$$= x^2 + x + 2$$

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

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11. Assume that $L : \mathcal{P}_2 \mapsto \mathcal{P}_2$ is the linear transformation that is described by the matrix M below relative to the ordered bases $\mathcal{B}_1 = \{1, x, x^2\}$ for the domain and $\mathcal{B}_2 = \{1, (x+1), (x+1)^2\}$ for the target space

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Find scalars a, b, and c such that

$$L(a + bx + cx^2) = 1 + 2x.$$

$= 2(x+1) - 1$

7 pts.

$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - 2 + 0 \\ 0 + 2 + 0 \\ 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$$

$-3(1) + 2(x) + 0(x^2) = -3 + 2x$

$\therefore a = -3 \quad b = 2 \quad c = 0$

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12. Find scalars a, b, and c that the set $\mathcal{B}$ formed by the following vectors is an orthogonal basis for $\mathbb{R}^3$.

$$\mathcal{B} = \{[1,1,-3]^t, [1,2,a]^t, [b,c,1]^t\}$$

6 pts.

$[1,1,-3] \cdot [1,2,a] = 0$

$1 \cdot 1 + 1 \cdot 2 - 3 \cdot a = 0$

$1 + 2 - 3a = 0$

$a = 1$

$$\begin{cases} [1,1,-3] \cdot [b,c,1] = 0 \\ [1,2,1] \cdot [b,c,1] = 0 \end{cases}$$

$$\begin{cases} b + c - 3 = 0 \\ b + 2c + 1 = 0 \end{cases}$$

$$\begin{cases} b = 7 \\ c = -4 \end{cases}$$

$$\begin{cases} a = 1 \\ b = 7 \\ c = -4 \end{cases}$$

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13. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$ below.

$$\mathcal{B} = \{[1,1,1,1]^t, [1,1,1,-3]^t\}$$

(a) Find the projection of $Z = [1,2,1,1]^t$ to $\mathcal{W}$. *4 pts.*

let B_1, B_2 be columns of B.

$B_1 \cdot B_2 = 0$ $\therefore$ they are orthogonal

$\therefore \text{Proj}_Z W$

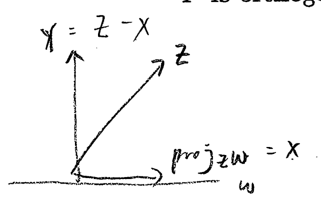
$$= \frac{Z \cdot B_1}{B_1 \cdot B_1} B_1 + \frac{Z \cdot B_2}{B_2 \cdot B_2} B_2$$

$$= \frac{1+2+1+1}{1+1+1+1} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1+2+1-3}{1+1+1+9} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \frac{5}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 15+1 \\ 15+1 \\ 15+1 \\ 15-3 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 16 \\ 16 \\ 16 \\ 12 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \\ 1 \end{bmatrix}$$

(b) Find vectors X and Y such that $Z = X + Y$ where $X \in \mathcal{W}$ and Y is orthogonal to $\mathcal{W}$. *3 pts.*



$\therefore X = \text{Proj}_Z W = \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \\ 1 \end{bmatrix}$

$Y = Z - X = \begin{bmatrix} 1 - 4/3 \\ 2 - 4/3 \\ 1 - 4/3 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 0 \end{bmatrix}$

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14. Let $\mathcal{W}$ be the subspace of $\mathbb{R}^4$ spanned by the set $\mathcal{B}$ below. We want to apply the Gram-Schmidt process to produce an orthogonal basis $\mathcal{P} = \{P_1, P_2, P_3\}$ for $\mathcal{W}$.

$$\mathcal{B} = \{[1,1,0,1]^t, [0,2,1,0]^t, [1,1,1,1]^t\}$$

(a) Compute the first two Gram-Schmidt basis elements P_1 and P_2. *3 pts.*

$$P_1 = B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P_2 = B_2 - \frac{B_2 \cdot P_1}{P_1 \cdot P_1} P_1$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{1+1+1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0-2 \\ 6-2 \\ 3-0 \\ 0-2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 4/3 \\ 1 \\ -2/3 \end{bmatrix}$$

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(b) Assume that your answer to part 14a was $P_1 = [1, -2, 0, 0]^t$ and $P_2 = [2, 1, 0, 1]^t$ (which is not correct). What would you obtain for P_3 if you continue to follow the Gram-Schmidt process using these incorrect answers? *5 pts.*

$$P_3 = B_3 - \frac{B_3 \cdot P_1}{P_1 \cdot P_1} P_1 - \frac{B_3 \cdot P_2}{P_2 \cdot P_2} P_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1-2}{1+4} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} - \frac{2+1+1}{4+1+1} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} 15 + 3 - 20 \\ 15 - 6 - 10 \\ 15 + 0 + 0 \\ 15 + 0 - 10 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} -2 \\ -1 \\ 15 \\ 5 \end{bmatrix} = \begin{bmatrix} -2/15 \\ -1/15 \\ 1 \\ 1/3 \end{bmatrix}.$$

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Theorem 5.5 on p.289

15. Prove the following result which is Theorem 2 on p. 294. Here it is assumed that all of the entries a_{ij} of A are real numbers and all of the λ_i are real numbers. *8 pts*

Theorem. *Let A be an $n \times n$ matrix and let $Q_1, Q_2, \ldots, Q_k$ be eigenvectors for A in $\mathbb{R}^n$ corresponding to the eigenvalues λ_i. Suppose that the λ_i are all different. Then $\{Q_1, Q_2, \ldots, Q_k\}$ is a linearly independent subset of $\mathbb{R}^n$.*

equation ① : $c_1Q_1 + c_2Q_2 + \cdots c_kQ_k = 0$

A① : $\lambda_1 c_1 Q_1 + \lambda_2 c_2 Q_2 + \cdots \lambda_k c_k Q_k = 0$

λ_k① : $\lambda_k c_1 Q_1 + \lambda_k c_2 Q_2 + \cdots \lambda_k c_k Q_k = 0$

equation ② : A① $- \lambda_k$① $= (\lambda_1 - \lambda_k) c_1 Q_1 + (\lambda_2 - \lambda_k) c_2 Q_2 \cdots (\lambda_{k-1} - \lambda_k) c_{k-1} Q_{k-1} = 0$

equation ③ : A② $- \lambda_{k-1}$② $= (\lambda_1 - \lambda_k)(\lambda_1 - \lambda_{k-1}) c_1 Q_1 + (\lambda_2 - \lambda_k)(\lambda_2 - \lambda_{k-1}) c_2 Q_2 + \cdots\cdots (\lambda_{k-2} - \lambda_k)(\lambda_{k-2} - \lambda_{k-1}) c_{k-2} Q_{k-2} = 0$

equation ④ : A③ $- \lambda_{k-2}$③ $= (\lambda_1 - \lambda_k)(\lambda_1 - \lambda_{k-1})(\lambda_1 - \lambda_{k-2}) c_1 Q_1 + \cdots +$
$(\lambda_{k-3} - \lambda_k)(\lambda_{k-3} - \lambda_{k-1})(\lambda_{k-3} - \lambda_{k-2}) c_{k-3} Q_{k-3} = 0$

repeat this process to equation $(k-1)$: $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdots (\lambda_1 - \lambda_k) c_1 Q_1 = 0$

$\because$ λ_i is different, Q_1 is eigen vector, $Q_1 \neq 0$

$\therefore c_1 = 0$

substitude it to equation $(k-2)$:

$(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4) \cdots (\lambda_1 - \lambda_k) c_1 Q_1 + (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) \cdots (\lambda_2 - \lambda_k) c_2 Q_2 = 0$

$\because c_1 = 0$

$\therefore (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) \cdots (\lambda_2 - \lambda_k) c_2 Q_2 = 0$

because of the similar reason, $c_2 = 0$

repeat this process to equation ①, get $c_1, c_2 \ldots c_k$ are all 0.

$\therefore$ they are linearly independent.

This proof is much TOO COMPLICATED. See the proof of Theorem5.5 on p. 289 of the text.